PETERSSON'S TRACE FORMULA AND THE HECKE EIGENVALUES OF HILBERT MODULAR FORMS

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Abstract. Using an explicit relative trace formula, we obtain a Petersson trace formula for holomorphic Hilbert modular forms. Our main result expresses a sum (over a Hecke eigenbasis) of products of Fourier coefficients and Hecke eigenvalues in terms of generalized Kloosterman sums and Bessel functions. As an application we show that the normalized Hecke eigenvalues for a fixed prime $\mathfrak{p}$ have an asymptotic weighted equidistribution relative to a polynomial times the Sato-Tate measure, as the norm of the level goes to ∞.

1. Introduction

Let $h \in S_{\mathtt{k}}(\Gamma_0(N))$ be a Hecke eigenform ($\mathtt{k}$ even), and for a prime $p \nmid N$ define the normalized Hecke eigenvalue ν_p^h by

$$p^{-\frac{\mathtt{k}-1}{2}} T_p h = \nu_p^h \, h.$$

The Ramanujan-Petersson conjecture asserts that $|\nu_p^h| \leq 2$. This is a theorem of Deligne. Because we have assumed trivial central character, the operator T_p is self-adjoint, so its eigenvalues are real numbers, and thus

$$\nu_p^h \in [-2, 2].$$

For a fixed non-CM newform h, the Sato-Tate conjecture predicts that the set $\{\nu_p^h : p \nmid N\}$ is equidistributed in $[-2, 2]$ relative to the Sato-Tate measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,dx & \text{when } x \in [-2,2], \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Taylor has recently proven this in many cases when $\mathtt{k} = 2$ [Ta].

When the prime p is fixed, the normalized eigenvalues of T_p on the space $S_{\mathtt{k}}(\Gamma_0(N))$ are asymptotically equidistributed relative to the measure

$$d\mu_p(x) = \frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2}\, d\mu_\infty(x)$$

as $\mathtt{k} + N \to \infty$. This was proven in the late 1990's by Serre [Se], and independently (for $N = 1$) by Conrey, Duke and Farmer [CDF]. An extension of the result (in the level aspect only) to Hilbert modular forms is given in [Li2]. These results have an antecedent in work of Bruggeman in 1978, who proved that the eigenvalues of T_p on Maass forms of level 1 exhibit the same distribution ([Br], Sect. 4; see also Sarnak [Sa]). Even earlier, in 1968 Birch proved a similar result for sizes of elliptic curves over $\mathbf{F}_p$ [Bi].

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In this paper we prove a different result on the asymptotic distribution of Hecke eigenvalues, valid over a totally real number field F. The following is a special case of a more general result (cf. Theorem 6.6). Notation and terminology will be defined precisely later on.

Theorem 1.1. *Let F be a totally real number field, and let m be a totally positive element of the inverse different $\mathfrak{d}^{-1} \subset F$. For a cusp form φ on $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)$ with trivial central character, let W_m^φ denote its m^{th} Fourier coefficient (see (10)). Define a weight*

$$w_\varphi = \frac{|W_m^\varphi(1)|^2}{\|\varphi\|^2}.$$

Fix a prime ideal $\mathfrak{p} \nmid m\mathfrak{d}$. Let $\{\varphi\}$ be an orthogonal basis of eigenforms of prime-to-$\mathfrak{p}$ level $\mathfrak{N}$ and weight $(\mathtt{k}_1, \ldots, \mathtt{k}_r)$, with all $\mathtt{k}_j > 2$ and even. For such φ, let

$$\nu_\mathfrak{p}^\varphi \in [-2, 2]$$

denote the associated normalized eigenvalue of the Hecke operator $T_\mathfrak{p}$. Then the w_φ-weighted distribution of the eigenvalues $\nu_\mathfrak{p}^\varphi$ is asymptotically uniform relative to the Sato-Tate measure as the norm of $\mathfrak{N}$ goes to ∞. This means that for any continuous function $f : \mathbf{R} \longrightarrow \mathbf{C}$,

$$\lim_{\mathbb{N}(\mathfrak{N})\to\infty} \frac{\sum_\varphi f(\nu_\mathfrak{p}^\varphi) w_\varphi}{\sum_\varphi w_\varphi} = \int_\mathbf{R} f(x) d\mu_\infty(x).$$

Noteworthy is the fact that here the measure and the weights are independent of $\mathfrak{p}$, in contrast to Serre's (unweighted) result. The Sato-Tate measure $d\mu_\infty$ is nonzero on any subinterval of $[-2, 2]$, so in particular the above illustrates the density of the Hecke eigenvalues in $[-2, 2]$.

The proof of the above theorem involves a trace formula which may be of independent interest. In a previous paper [KL1], we detailed the way in which the classical 1932 Petersson trace formula can be realized as an explicit relative trace formula, as a conceptual alternative to the usual method using Poincaré series. Here we extend the technique and result to cusp forms on $\mathrm{GL}_2(\mathbf{A}_F)$, where F is an arbitrary totally real number field. We work with spaces $A_\mathtt{k}(\mathfrak{N}, \omega)$ of holomorphic Hilbert cusp forms of weight $\mathtt{k} = (\mathtt{k}_1, \ldots, \mathtt{k}_r)$, all strictly greater than 2, on the Hecke congruence subgroups $\Gamma_0(\mathfrak{N})$. Our main result is Theorem 5.11 in which a sum (over a Hecke eigenbasis) of terms involving the associated Fourier coefficients, Petersson norms and eigenvalues of a Hecke operator $T_\mathfrak{n}$, is expressed in terms of generalized Kloosterman sums and Bessel functions.

The incorporation of Hecke eigenvalues is a novel feature of this generalized sum formula. In Section 6 we prove a more general version of the above weighted distribution theorem by an argument based on bounding the terms of the sum formula in terms of $\mathbb{N}(\mathfrak{N})$, following the technique for the $F = \mathbf{Q}$ case from [Li1]. This is analogous to the way in which Serre's result follows from bounding the terms of the Eichler-Selberg formula for $\mathrm{tr}(T_{p^m})$.

In Section 7 we give some variants of the generalized Petersson formula, including a simplified version for the case where F has narrow class number 1. In Corollary 7.3, setting $\mathfrak{n} = 1$ for the trivial Hecke operator, we recover the classical extension of Petersson's formula to Hilbert modular forms, as follows from a 1954 paper of Gundlach, [Gu]. His results are valid for cusp forms of uniform weight $\mathtt{k} \geq 2$. Refer to §2 of [Lu] for an overview of the classical derivation.

In 1980 Kuznetsov gave an analog of Petersson's formula involving Maass forms and Eisenstein series on the spectral side, [Ku]. As an application, he used his formula to give estimates for sums of Kloosterman sums. His work was reformulated in a representation-theoretic setting by various authors, see [CPS] and its references, notably [MW] where the general rank-1 case is treated. More recently Bruggeman, Miatello and Pacharoni gave a general Kuznetsov trace formula for automorphic forms on $\mathrm{SL}_2(F_\infty)$ of uniform even weight ([BMP], Theorem 2.7.1). An important application of their formula is an estimate for sums of Kloosterman sums.

In contrast to the above results, here we are concerned only with holomorphic cusp forms, i.e. those whose infinity types are discrete series. For our test function, we take discrete series matrix coefficients at the infinite components. This serves to isolate the holomorphic part of the cuspidal spectrum. At the finite places we take Hecke operators, which introduces Hecke eigenvalues into the final formula.

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2. General setting

We recall the general setting of Jacquet's relative trace formula [Ja]. Full details for the discussion below are given in §2 of [KL1]. Let F be a number field, with adele ring $\mathbf{A}$. Let G be a reductive algebraic group defined over F. Let H be an F-subgroup of $G \times G$, with $H(\mathbf{A})$ unimodular. We assume that $H(F)\backslash H(\mathbf{A})$ is compact. Define a right action of H on G by $g(x, y) = x^{-1}gy$. For $g \in G$, let H_g be the stabilizer of g, i.e.

$$H_g = \{(x, y) \in H \,|\, x^{-1}gy = g\}.$$

For $\delta \in G(F)$, let $[\delta]$ be the $H(F)$-orbit of δ in $G(F)$, i.e.

$$[\delta] = \{x^{-1}\delta y \,|\, (x, y) \in H(F)\}.$$

Each element of $[\delta]$ can be expressed uniquely in the form $u^{-1}\delta v$ for some $(u, v) \in H_\delta(F)\backslash H(F)$.

Let f be a continuous function on $G(\mathbf{A})$, and let

$$K(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \quad (x, y \in G(\mathbf{A})) \tag{2}$$

be the associated kernel function. We assume that the above sum is uniformly absolutely convergent on compact subsets of $H(\mathbf{A})$. In particular, $K(x, y)$ is a continuous function on the compact set $H(F)\backslash H(\mathbf{A})$. Let $\chi(x, y)$ be a character of $H(\mathbf{A})$, invariant under $H(F)$. Consider the expression

$$\int_{H(F)\backslash H(\mathbf{A})} K(x, y)\chi(x, y)d(x, y), \tag{3}$$

where $d(x, y)$ is a Haar measure on $H(\mathbf{A})$. A relative trace formula results from computing this integral using spectral and geometric expressions for $K(x, y)$. Using the geometric expression (2), it is straightforward to show that the integral (3) is equal to $\sum_{[\delta]} I_\delta(f)$, where

$$I_\delta(f) = \int_{H_\delta(F)\backslash H(\mathbf{A})} f(x^{-1}\delta y)\chi(x, y)d(x, y).$$

We see that

$$I_\delta(f) = \int_{H_\delta(\mathbf{A})\backslash H(\mathbf{A})} f(x^{-1}\delta y) \left[\int_{H_\delta(F)\backslash H_\delta(\mathbf{A})} \chi(rx, sy) d(r,s)\right] d(x,y).$$

An orbit $[\delta]$ is **relevant** if χ is trivial on $H_\delta(\mathbf{A})$. From the above, we see that $I_\delta = 0$ whenever $[\delta]$ is not relevant. Indeed the expression in the brackets is equal to $\chi(x_0, y_0)\int_{H_\delta(F)\backslash H_\delta(\mathbf{A})} \chi(r,s)d(r,s)$, where $(x_0, y_0) \in H(\mathbf{A})$ is any representative for (x, y), and this integral vanishes unless δ is relevant.

3. Preliminaries

3.1. **Notation.** Throughout this paper we work over a totally real number field $F \neq \mathbf{Q}$. Let

$$r = [F : \mathbf{Q}],$$

and let $\sigma_1, \dots, \sigma_r$ be the distinct embeddings $F \hookrightarrow \mathbf{R}$. Let $\infty_1, \dots, \infty_r$ denote the corresponding archimedean valuations. Let $\mathcal{O}$ be the ring of integers of F. We will generally use Gothic letters $\mathfrak{a}, \mathfrak{b}$ etc. to denote fractional ideals of F. We reserve $\mathfrak{p}$ for prime ideals. Let $v = v_\mathfrak{p}$ be the discrete valuation corresponding to $\mathfrak{p}$. Let $\mathcal{O}_v$ be the ring of integers in the local field F_v, and let $\varpi_v \in \mathcal{O}_v$ be a generator of the maximal ideal $\mathfrak{p}_v = \mathfrak{p}\mathcal{O}_v$.

Let $\mathrm{N} : F \to \mathbf{Q}$ denote the norm map. For a nonzero ideal $\mathfrak{a} \subset \mathcal{O}$, let $\mathbb{N}(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$ denote the absolute norm. This extends by multiplicativity to the group of nonzero fractional ideals of F. For $\alpha \in F^*$, we define

$$\mathbb{N}(\alpha) = \mathbb{N}(\alpha\mathcal{O}) = |\,\mathrm{N}(\alpha)|.$$

We also use the above norms in the local setting with the analogous meanings. We normalize the absolute value on F_v by $|\varpi_v|_v = \mathbb{N}(\mathfrak{p})^{-1}$.

By Dirichlet's unit theorem, the unit group of F is

$$\mathcal{O}^* \cong (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}^{r-1}.$$

Letting $\mathcal{O}^{*2}$ denote the subgroup consisting of squares of units, we see that $\mathcal{O}^*/\mathcal{O}^{*2} \cong (\mathbf{Z}/2\mathbf{Z})^r$ is a finite group. Let

$$U = \{u_1, \dots, u_{2^r}\} \subset \mathcal{O}^* \tag{4}$$

be a fixed set of representatives for $\mathcal{O}^*/\mathcal{O}^{*2}$.

For a fractional ideal $\mathfrak{a} \subset F$, let $\mathrm{ord}_v\,\mathfrak{a}$ (or $\mathrm{ord}_\mathfrak{p}\,\mathfrak{a}$) denote the order. Let $\mathfrak{a}_v$ (or $\mathfrak{a}_\mathfrak{p}$) denote its localization, so $\mathfrak{a}_v = \varpi_v^{\mathrm{ord}_v(\mathfrak{a})}\mathcal{O}_v$. Let $\mathbf{A}$ denote the adele ring of F, with finite adeles $\mathbf{A}_{\mathrm{fin}}$, so that

$$\mathbf{A} = F_\infty \times \mathbf{A}_{\mathrm{fin}},$$

where $F_\infty = F \otimes \mathbf{R} \cong \mathbf{R}^r$. Let $\widehat{\mathcal{O}} = \prod_{v<\infty} \mathcal{O}_v$. Generally if $\mathfrak{a}$ is a fractional ideal of F, we write $\widehat{\mathfrak{a}} = \mathfrak{a}\widehat{\mathcal{O}} = \prod_{v<\infty} \mathfrak{a}_v \subset \mathbf{A}_{\mathrm{fin}}$. Let $\mathrm{Cl}(F)$ be the class group of F, of cardinality $h(F)$. For a fractional ideal $\mathfrak{a}$, let $[\mathfrak{a}]$ represent its image in the class group.

Let F^+ denote the set of totally positive elements of F. We let F^+_∞ denote the subset of F_∞ of vectors whose entries are all positive. Let

$$\mathfrak{d}^{-1} = \{x \in F |\ \mathrm{tr}^F_\mathbf{Q}(x\mathcal{O}) \subset \mathbf{Z}\}$$

denote the inverse different. We set $\mathfrak{d}^{-1}_+ = \mathfrak{d}^{-1} \cap F^+$.

We use the mathtt font to represent finite ideles. Thus we write

$$\mathtt{a} \in \mathbf{A}_{\text{fin}}^*, \quad \widehat{\mathfrak{a}} = \mathtt{a}\widehat{\mathcal{O}}, \quad \mathfrak{a} = \mathcal{O} \cap \widehat{\mathfrak{a}}. \tag{5}$$

In the other direction, given a fractional ideal $\mathfrak{a} \subset F$, there is an element $\mathtt{a} \in \mathbf{A}_{\text{fin}}^*$ such that (5) holds. Explicitly, we can take $\mathtt{a}_v = \varpi_v^{\text{ord}_v \mathfrak{a}}$. The element $\mathtt{a}$ is unique up to $\widehat{\mathcal{O}}^*$. We define norms of ideles by taking the products of the local norms. For example, in the situation of (5), we have

$$|\mathtt{a}|_{\text{fin}} = \prod_{v<\infty} |\mathtt{a}_v|_v = \prod_{v<\infty} \mathbb{N}(\mathtt{a}_v)^{-1} = \mathbb{N}(\mathtt{a})^{-1} = \mathbb{N}(\mathfrak{a})^{-1}.$$

We use Roman or Greek letters for rational elements $a, \alpha \in F$.

3.2. **Haar measure.** We use Lebesgue measure on $\mathbf{R}$, and take the product measure on $F_\infty \cong \mathbf{R}^r$. We normalize Haar measure on each non-archimedean completion F_v by taking $\text{meas}(\mathcal{O}_v) = 1$. This choice induces a Haar measure on $\mathbf{A}_{\text{fin}}$ with $\text{meas}(\widehat{\mathcal{O}}) = 1$, and because $\mathbf{A} = F + F_\infty \times \widehat{\mathcal{O}}$,

$$\begin{aligned} \text{meas}(F\backslash\mathbf{A}) &= \text{meas}((F \cap \widehat{\mathcal{O}})\backslash(F_\infty \times \widehat{\mathcal{O}})) = \text{meas}((\mathcal{O}\backslash F_\infty) \times \widehat{\mathcal{O}}) \\ &= \text{meas}(\mathcal{O}\backslash F_\infty) = d_F^{1/2}, \end{aligned} \tag{6}$$

where d_F is the discriminant of F. The resulting measure on $\mathbf{A}$ is not self-dual.

Let $G = \text{GL}_2$. We normalize Haar measure on $G(\mathbf{R})$ as follows. Use Lebesgue measure dx to define a measure dn on the unipotent subgroup $N(\mathbf{R}) \cong \mathbf{R}$. On $\mathbf{R}^*$ we use the measure $dx/|x|$. We take the product measure dm on the diagonal subgroup $M(\mathbf{R}) \cong \mathbf{R}^* \times \mathbf{R}^*$, and normalize dk on $\text{SO}_2(\mathbf{R})$ to have total measure 1. On $G(\mathbf{R}) = M(\mathbf{R})N(\mathbf{R})\,\text{SO}_2(\mathbf{R})$ we take $dg = dm\,dn\,dk$.

Let

$$K_{\text{fin}} = \prod_{v<\infty} K_v = \prod_{v<\infty} G(\mathcal{O}_v) = G(\widehat{\mathcal{O}})$$

be the standard maximal compact subgroup of $G(\mathbf{A}_{\text{fin}})$. We normalize Haar measure on $G(\mathbf{A}_{\text{fin}})$ by taking $\text{meas}(K_v) = 1$ for all $v < \infty$. We let Z denote the center of G, and write

$$\overline{G} = G/Z.$$

We normalize Haar measure on $\overline{G}(F_v)$ by taking $\text{meas}(\overline{K}_v) = 1$.

3.3. **Hilbert modular forms.** Let $\mathfrak{N} \subset \mathcal{O}$ be an integral ideal of F. Let $\mathtt{k} = (\mathtt{k}_1, \ldots, \mathtt{k}_r)$ be an r-tuple of positive integers, each greater than or equal to 2. Let

$$\omega : F^*\backslash\mathbf{A}^* \to \mathbf{C}^*$$

be a unitary Hecke character. Write $\omega = \prod_v \omega_v$. We assume that:

(1) the conductor of ω divides $\mathfrak{N}$
(2) $\omega_{\infty_j}(x) = \text{sgn}(x)^{\mathtt{k}_j}$ for all $1 \le j \le r$.

The first condition means that ω_v is trivial on $1 + \mathfrak{N}_v$ for all $v|\mathfrak{N}$, and unramified for all $v \nmid \mathfrak{N}$. In other words, ω_{fin} is trivial on the open set

$$\widehat{\mathcal{O}}_{\mathfrak{N}}^* \overset{\text{def}}{=} \widehat{\mathcal{O}}^* \cap (1 + \mathfrak{N}\widehat{\mathcal{O}}) = \prod_{v|\mathfrak{N}} (1 + \mathfrak{N}_v) \prod_{v\nmid\mathfrak{N}} \mathcal{O}_v^*.$$

Thus ω can be viewed as a character of the ray class group mod $\mathfrak{N}$

$$\text{Cl}(\mathfrak{N}) \cong \mathbf{A}^*/[F^*(F_\infty^+ \times \widehat{\mathcal{O}}_{\mathfrak{N}}^*)].$$

We freely identify $Z(\mathbf{A})$ with $\mathbf{A}^*$ throughout this paper. For example if $z \in Z(\mathbf{A})$ we write $\omega(z)$. Let $G = \mathrm{GL}_2$ and let

$$L^2(\omega) = L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}), \omega)$$

be the space of left $G(F)$-invariant functions on $G(\mathbf{A})$ which transform by ω under the center and which are square integrable over $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$. Let R denote the right regular representation of $G(\mathbf{A})$ on $L^2(\omega)$. Let $L^2_0(\omega)$ be the subspace of cuspidal functions. We know that the restriction of R to $L^2_0(\omega)$ decomposes as a discrete sum of irreducible representations. These are the cuspidal representations π. Every such π factorizes as a restricted tensor product of admissible local representations $\otimes\pi_v$.

Define groups

$$K_0(\mathfrak{N}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\mathrm{fin}} |\, c \in \mathfrak{N}\widehat{\mathcal{O}}\}$$

and

$$K_1(\mathfrak{N}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{N}) |\, d \in 1 + \mathfrak{N}\widehat{\mathcal{O}}\}.$$

These are open compact subgroups of $G(\mathbf{A}_{\mathrm{fin}})$. Let

$$H_{\mathtt{k}}(\mathfrak{N}, \omega) = \bigoplus \pi, \tag{7}$$

where π runs through all cuspidal representations in $L^2_0(\omega)$ for which:

(1) $\pi_{\mathrm{fin}} = \otimes_{v<\infty}\pi_v$ contains a nonzero $K_1(\mathfrak{N})$-fixed vector
(2) $\pi_{\infty_j} = \pi_{\mathtt{k}_j}$ is the discrete series representation of $G(\mathbf{R})$ of weight $\mathtt{k}_j$, for $j = 1, \dots, r$.

The central character of $\pi_{\mathtt{k}_j}$ is given by

$$\chi_{\pi_{\mathtt{k}_j}}(x) = \mathrm{sgn}(x)^{\mathtt{k}_j} = \omega_{\infty_j}(x).$$

For a discrete series representation π_{∞_j}, let $v_{\pi_{\infty_j}}$ be a lowest weight vector, unique up to nonzero multiples. Define the subspace

$$A_{\mathtt{k}}(\mathfrak{N}, \omega) = \bigoplus_{\pi} \mathbf{C} v_{\pi_{\infty_1}} \otimes \dots \otimes v_{\pi_{\infty_r}} \otimes \pi_{\mathrm{fin}}^{K_1(\mathfrak{N})}, \tag{8}$$

where π runs through all irreducible summands of $H_{\mathtt{k}}(\mathfrak{N}, \omega)$. Here $\pi_{\mathrm{fin}}^{K_1(\mathfrak{N})}$ is the subspace of $K_1(\mathfrak{N})$-fixed vectors in the space of π_{fin}.

Proposition 3.1. *The space $A_{\mathtt{k}}(\mathfrak{N}, \omega)$ defined above is equal to the set of $\varphi \in L^2_0(\omega)$ satisfying:*

(1) $\varphi(gk_{\mathrm{fin}}) = \varphi(g)$ *for all* $k_{\mathrm{fin}} \in K_1(\mathfrak{N})$
(2) $\varphi(gk_\infty) = \prod_{j=1}^{r} e^{i\mathtt{k}_j\theta_j}\varphi(g)$ *for all* $k_\infty = \prod_j k_{\theta_j} \in K_\infty = \mathrm{SO}_2(\mathbf{R})^r$
(3) *For any fixed $x \in G(\mathbf{A})$ and $1 \le j \le r$, the function $g_{\infty_j} \mapsto \varphi(xg_{\infty_j})$ is annihilated by $R(E^-)$, where $E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathfrak{gl}_2(\mathbf{C})$ and R denotes the right regular action of $G(F_{\infty_j})$.*

The elements of $A_{\mathtt{k}}(\mathfrak{N}, \omega)$ are continuous functions on $G(\mathbf{A})$.

Proof. The proof is the same as for the case $F = \mathbf{Q}$ given in Theorem 12.6 of [KL2]. For the continuity, see Lemma 3.3 of [Li2]. □

By a general theorem of Harish-Chandra, the space $A_{\mathrm{k}}(\mathfrak{N},\omega)$ is finite-dimensional (cf. [HC], [BJ]). In particular the set of π in (7) is finite.

3.4. **Fourier coefficients.** Let $\theta : \mathbf{A} \longrightarrow \mathbf{C}^*$ be the standard character of $\mathbf{A}$. Explicitly,

(1) $\theta_\infty(x) = e^{-2\pi i(x_1+\cdots+x_r)}$ for $x = (x_1,\ldots,x_r) \in F_\infty$
(2) For $v < \infty$, θ_v is the composition

$$\theta_v : F_v \xrightarrow{\mathrm{tr}^{F_v}_{\mathbf{Q}_p}} \mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \hookrightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{e^{(2\pi i\cdot)}} \mathbf{C}^*.$$

Note that θ_v is trivial precisely on the local inverse different $\mathfrak{d}_v^{-1} = \mathfrak{d}^{-1}\mathcal{O}_v = \{x \in F_v |\ \mathrm{tr}^{F_v}_{\mathbf{Q}_p}(x) \in \mathbf{Z}_p\}$. Recall that θ is trivial on F, and that every character on $F\backslash\mathbf{A}$ is of the form

$$\theta_m(x) = \theta(-mx)$$

for some $m \in F$. This identifies the discrete group F with the dual group $\widehat{F\backslash\mathbf{A}}$.

Consider the unipotent subgroup $N = \{\left(\begin{smallmatrix}1 & *\\ & 1\end{smallmatrix}\right)\}$ of G. As topological groups, $N(\mathbf{A}) \cong \mathbf{A}$, so characters of the two can be identified. For $m \in F$, we identify θ_m with a character on $N(F)\backslash N(\mathbf{A})$ in the obvious way.

For any smooth cusp form φ on $G(\mathbf{A})$ and $g \in G(\mathbf{A})$, the map $n \mapsto \varphi(ng)$ is a continuous function on $N(F)\backslash N(\mathbf{A})$, with a Fourier expansion

$$\varphi(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g) = \frac{1}{d_F^{1/2}} \sum_{m\in F} W_m^\varphi(g)\theta_m(x). \tag{9}$$

The coefficients are Whittaker functions defined by

$$W_m^\varphi(g) = \int_{F\backslash\mathbf{A}} \varphi(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g)\theta(mx)dx. \tag{10}$$

The purpose of the factor of $d_F^{-1/2}$ in (9) is to balance the fact that by our choice, $\mathrm{meas}(F\backslash\mathbf{A}) = d_F^{1/2}$. This non-selfdual measure is convenient for the calculations later on.

Let $\mathrm{y}_1,\ldots,\mathrm{y}_h \in \mathbf{A}^*_{\mathrm{fin}}$ be representatives for $\mathbf{A}^*_{\mathrm{fin}}/F^*\widehat{\mathcal{O}}^*$, the class group of F. Then

$$G(\mathbf{A}) = \bigcup_{i=1}^h G(F)\left[B(F_\infty)^+K_\infty \times \begin{pmatrix}\mathrm{y}_i & \\ & 1\end{pmatrix} K_1(\mathfrak{N})\right]$$

([Hi], §9.1). Here B denotes the subgroup of invertible upper triangular matrices. It follows that an element $\varphi \in A_{\mathrm{k}}(\mathfrak{N},\omega)$ is determined by the values

$$\varphi(\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)_\infty \left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)_\infty \left(\begin{smallmatrix}\mathrm{y}_i & \\ & 1\end{smallmatrix}\right)_{\mathrm{fin}})$$

for $y \in F_\infty^+$ and $x \in F_\infty$. We define for $y \in \mathbf{A}^*$ the notation

$$W_m^\varphi(y) = W_m^\varphi(\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)). \tag{11}$$

By the above remarks, φ is determined by the coefficients $W_m^\varphi(y)$ for $y \in \mathbf{A}^*$.

Proposition 3.2. *For $\varphi \in A_{\mathrm{k}}(\mathfrak{N},\omega)$,*

- $W_m^\varphi(g) = W_1^\varphi(\left(\begin{smallmatrix}m & \\ & 1\end{smallmatrix}\right)g)$ *for all $m \in F^*$ and $g \in G(\mathbf{A})$*
- *For any $\mathrm{u} \in \widehat{\mathcal{O}}^*$ and $y \in \mathbf{A}^*$, $W_m^\varphi(y) = W_m^\varphi(\mathrm{u}y)$ for all $m \in F^*$*
- *For $\mathrm{y} \in \mathbf{A}^*_{\mathrm{fin}}$, $W_1^\varphi(\mathrm{y}) \neq 0$ only if $\mathrm{y} \in \widehat{\mathfrak{d}}^{-1}$.*

(Here and throughout, we identify $\mathbf{A}^*_{\mathrm{fin}}$ with $\{1_\infty\} \times \mathbf{A}^*_{\mathrm{fin}} \subset \mathbf{A}^*$.)

Proof. These are standard facts. The first follows by a change of variables in (10) plus the left $G(F)$-invariance of φ. The $K_1(\mathfrak{N})$-invariance of φ gives the second, and also implies that for $\mathtt{u} \in \widehat{\mathcal{O}}$

$$\varphi(\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathtt{y} & \\ & 1\end{smallmatrix}\right)) = \varphi(\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathtt{y} & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \mathtt{u}\\ & 1\end{smallmatrix}\right)) = \varphi(\left(\begin{smallmatrix}1 & x+\mathtt{uy}\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathtt{y} & \\ & 1\end{smallmatrix}\right)).$$

This means that $W_m^\varphi(\mathtt{y}) = W_m^\varphi(\mathtt{y})\theta(-m\mathtt{uy})$ for all $\mathtt{u} \in \widehat{\mathcal{O}}$, which gives the third when $m = 1$. (See also §9.1 of [Hi].) □

Proposition 3.3. *For any* $\varphi \in A_{\mathrm{k}}(\mathfrak{N}, \omega)$ *and any* $y \in F_\infty^+$, $W_m^\varphi(y) = 0$ *unless* $m \in \mathfrak{d}_+^{-1} = \mathfrak{d}^{-1} \cap F^+$.

Proof. By the definition of cuspidality, the constant term $\varphi_N(g)$ vanishes for a.e. $g \in G(\mathbf{A})$. Because φ is actually continuous, it follows that $\varphi_N(g) = 0$ for all g. Therefore when $m = 0$,

$$W_0^\varphi(g) = \varphi_N(g) = 0. \tag{12}$$

To φ we can attach a holomorphic function on $\mathbf{H}^r$ by

$$h(x+iy) = \varphi(\left(\begin{smallmatrix}y & x\\ & 1\end{smallmatrix}\right)_\infty \times 1_{\mathrm{fin}}) \prod_{j=1}^r y_j^{-\mathrm{k}_j/2}. \tag{13}$$

Then h is a Hilbert modular form for the group $\Gamma_1(\mathfrak{N}) = \mathrm{SL}_2(F) \cap K_1(\mathfrak{N})$, so it has a Fourier expansion of the form

$$h(x+iy) = a_0(h) + \sum_{m \in \mathfrak{d}_+^{-1}} a_m(h) e^{-2\pi\,\mathrm{tr}(my)} \theta_{\infty,m}(x),$$

where $\mathrm{tr}(my) = \sum_{j=1}^r \sigma_j(m) y_j$. For any $y \in F_\infty^+$, the Fourier coefficients $a_m(h)$ and $W_m^\varphi(y)$ are related by

$$W_m^\varphi(y) = d_F^{1/2}(\prod_{j=1}^r y_j^{\mathrm{k}_j/2}) e^{-2\pi\,\mathrm{tr}(my)} a_m(h). \tag{14}$$

This follows immediately by equating the classical and adelic Fourier expansions of $\varphi(\left(\begin{smallmatrix}y_\infty & x_\infty\\ 0 & 1\end{smallmatrix}\right))$. Together with (12), this implies that $W_m^\varphi(y) = 0$ unless $m \in \mathfrak{d}_+^{-1}$. □

4. Construction of the test function

The right regular action of an element $f \in L^1(G(\mathbf{A}), \omega^{-1})$ on $L^2(\omega)$ is given by

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(g)\phi(xg)dg. \tag{15}$$

In this section we will construct a continuous integrable function f such that $R(f)$ has finite rank (vanishing on $A_{\mathrm{k}}(\mathfrak{N}, \omega)^\perp$), and acts like a Hecke operator on $A_{\mathrm{k}}(\mathfrak{N}, \omega)$. The function will be defined locally:

$$f = f_\infty f_{\mathrm{fin}} = \prod_{j=1}^r f_{\infty_j} \prod_{v<\infty} f_v.$$

4.1. **Archimedean test functions.** For $j = 1, \dots, r$ let $v_{\mathtt{k}_j}$ be a lowest weight unit vector for $\pi_{\mathtt{k}_j}$ and let $d_{\mathtt{k}_j}$ be the formal degree of $\pi_{\mathtt{k}_j}$ relative to the measure on $G(\mathbf{R})$ fixed in Sect. 3.2. We take $f_{\infty_j}(g)$ to be the normalized matrix coefficient $d_{\mathtt{k}_j}\overline{\langle \pi_{\mathtt{k}_j}(g)v_{\mathtt{k}_j}, v_{\mathtt{k}_j}\rangle}$. Explicitly, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then (cf. [KL2], Theorem 14.5)

$$f_{\infty_j}(g) = \begin{cases} \dfrac{(\mathtt{k}_j - 1)}{4\pi} \dfrac{\det(g)^{\mathtt{k}_j/2}(2i)^{\mathtt{k}_j}}{(-b + c + (a+d)i)^{\mathtt{k}_j}} & \text{if } \det(g) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

This function is integrable over $\overline{G}(\mathbf{R})$ if and only if $\mathtt{k}_j > 2$. Therefore in order for (15) to converge we must assume henceforth that

$$\mathtt{k}_j > 2 \quad (j = 1, \dots, r).$$

Proposition 4.1. *Let $f_\infty = \prod_j f_{\infty_j}$ be as above, and suppose f_{fin} is a bi-$K_1(\mathfrak{N})$-invariant function on $G(\mathbf{A}_{\text{fin}})$ satisfying $f_{\text{fin}}(zg) = \omega_{\text{fin}}(z)^{-1} f_{\text{fin}}(g)$ and whose support is compact mod $Z(\mathbf{A}_{\text{fin}})$. Then $R(f)$ vanishes on $A_{\mathtt{k}}(\mathfrak{N}, \omega)^\perp$ (the orthogonal complement in $L^2(\omega)$) and its image is a subspace of $A_{\mathtt{k}}(\mathfrak{N}, \omega)$.*

Proof. The case $F = \mathbf{Q}$ is proven in Corollary 13.13 of [KL2], and the general case is no different. The main point is that $\int_{N(\mathbf{R})} f_{\infty_j}(g_1 n g_2) dn = 0$ for each j. Using this, one shows that $R(f)\phi$ is cuspidal for each $\phi \in L^2(\omega)$. By the left $K_1(\mathfrak{N})$-invariance of f_{fin}, it is easy to see that $R(f)\phi$ is $K_1(\mathfrak{N})$-invariant, while the matrix coefficients project onto the span of the lowest weight vectors of the discrete series of the appropriate weight. Thus $R(f)\phi \in A_{\mathtt{k}}(\mathfrak{N}, \omega)$. (See also [Li2], Prop. 2.2). □

4.2. **Non-Archimedean test functions.** We now specify the local factors of f more precisely. Fix a discrete valuation v of F, and let $\mathfrak{p}$ be the corresponding prime ideal of $\mathcal{O}$. Let $\mathfrak{N} \subset \mathcal{O}$ be the ideal fixed earlier, and let $\mathfrak{N}_v = \mathfrak{N}\mathcal{O}_v$ be its localization.

Let $G_v = G(F_v)$, and similarly for its subgroups $Z_v = Z(F_v)$, etc. Suppose $f_v : G_v \to \mathbf{C}$ is a locally constant function whose support is compact modulo Z_v. Then the Hecke operator $R(f_v)$ is defined by

$$R(f_v)\phi(x) = \int_{\overline{G}_v} f_v(g)\phi(xg)dg$$

for any continuous function ϕ on G_v. Note that the integrand is not always well-defined modulo Z_v. In our situation, ϕ will be a function satisfying $\phi(zg) = \omega_v(z)\phi(g)$ for all $z \in Z_v$ and $g \in G_v$. Therefore we must require f_v to transform under the center by ω_v^{-1}.

The Hecke algebra of bi-$K_1(\mathfrak{N})_v$-invariant functions is generated by functions supported on sets of the form

$$Z_v K_1(\mathfrak{N})_v \, x_v \, K_1(\mathfrak{N})_v$$

for $x_v \in G_v$.

Fix an integral ideal $\mathfrak{n}_v$ in $\mathcal{O}_v$. We assume that $\mathfrak{n}_v$ is coprime to $\mathfrak{N}_v$, i.e., that either $\mathfrak{n}_v$ or $\mathfrak{N}_v$ is equal to $\mathcal{O}_v$. Define a set

$$M(\mathfrak{n}_v, \mathfrak{N}_v) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_v) | \, c \in \mathfrak{N}_v, \, (ad - bc)\mathcal{O}_v = \mathfrak{n}_v\}.$$

The determinant condition is equivalent to $\det g \in \varpi_v^{\mathrm{ord}_v(\mathfrak{n}_v)}\mathcal{O}_v^*$. By the Cartan decomposition of G_v we have

$$M(\mathfrak{n}_v, \mathfrak{N}_v) = \begin{cases} \displaystyle\bigcup_{\substack{i+j=\mathrm{ord}_v(\mathfrak{n}_v)\\ i\geq j\geq 0}} K_v \begin{pmatrix} \varpi_v^i & \\ & \varpi_v^j \end{pmatrix} K_v & \text{if } v \nmid \mathfrak{N} \\ \\ K_0(\mathfrak{N})_v & \text{if } v|\mathfrak{N}. \end{cases}$$

Clearly this is a compact set. We need to define a bi-$K_1(\mathfrak{N})_v$-invariant function $f_{\mathfrak{n}_v}$, supported on

$$Z_v M(\mathfrak{n}_v, \mathfrak{N}_v),$$

and with central character ω_v^{-1}. If $v|\mathfrak{N}$, for $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(\mathfrak{N})_v$ define

$$\omega_v(k) = \omega_v(d).$$

Because $c \in \mathfrak{N}_v$, one easily sees that this is a character of $K_0(\mathfrak{N})_v$. Now for $z \in Z_v$ and $m \in M(\mathfrak{n}_v, \mathfrak{N}_v)$, define

$$f_{\mathfrak{n}_v}(zm) = \begin{cases} \omega_v(z)^{-1} & \text{if } v \nmid \mathfrak{N} \\ \\ \psi(\mathfrak{N}_v)\omega_v(z)^{-1}\omega_v(m)^{-1} & \text{if } v|\mathfrak{N}. \end{cases} \tag{17}$$

Here, when $\mathfrak{p}|\mathfrak{N}$,

$$\psi(\mathfrak{N}_v) = \mathrm{meas}(\overline{K_1(\mathfrak{N})_v})^{-1} = [K_v : K_0(\mathfrak{N})_v] = \mathbb{N}(\mathfrak{p})^{\mathrm{ord}_v(\mathfrak{N})}(1 + \mathbb{N}(\mathfrak{p})^{-1}).$$

It is straightforward to show that $f_{\mathfrak{n}_v}$ is well-defined.

Lemma 4.2. *Suppose $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(F_v)$ and that $(\det g) = \mathfrak{n}_v$. Then $f_{\mathfrak{n}_v}(g) \neq 0$ if and only if $g \in M_2(\mathcal{O}_v)$ and $c \in \mathfrak{N}_v$.*

Proof. Note that $f_{\mathfrak{n}_v}(g) \neq 0$ if and only if $g = zm$, with $z \in Z(F_v)$, $m \in M(\mathfrak{n}_v, \mathfrak{N}_v)$. Taking determinants we see that z is a unit in $\mathcal{O}_v$ (identifying Z_v with F_v^*). Thus z can be absorbed into m, so in fact $g \in M(\mathfrak{n}_v, \mathfrak{N}_v)$ as required. □

Proposition 4.3. *The adjoint of the operator $R(f_{\mathfrak{p}_v^\ell})$ on $L^2(G_v, \omega_v)$ is given by*

$$R(f_{\mathfrak{p}_v^\ell})^* = \omega_v(\varpi_v^\ell)^{-1} R(f_{\mathfrak{p}_v^\ell}).$$

Proof. We have $R(f_{\mathfrak{p}_v^\ell})^* = R(f^*_{\mathfrak{p}_v^\ell})$ where $f^*_{\mathfrak{p}_v^\ell}(g) = \overline{f_{\mathfrak{p}_v^\ell}(g^{-1})}$. If $g = zm$, then

$$g^{-1} = z^{-1}m^{-1} = (z^{-1}\varpi_v^{-\ell})(\varpi_v^\ell m^{-1}) \in Z_v M(\mathfrak{n}_v, \mathfrak{N}_v).$$

The proposition follows easily from this. (Note that $\ell = 0$ if $v|\mathfrak{N}$.) □

When $v \nmid \mathfrak{N}$, the functions $f_{\mathfrak{n}_v}$ defined above linearly span the spherical Hecke algebra of bi-K_v-invariant complex-valued functions on G_v with central character ω_v^{-1}.

Now suppose $v \nmid \mathfrak{N}$ and write $\mathfrak{n}_v = \mathfrak{p}_v^\ell$ for $\ell > 0$. Let $\chi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \chi_1(a)\chi_2(d)$ be an unramified character of the Borel subgroup $B(F_v)$, and let (π, V_χ) be the representation of G_v obtained from χ by normalized induction. We assume that

$$\chi_1(z)\chi_2(z) = \omega_v(z)$$

for all $z \in Z_v$. Define a function $\phi_0 \in V_\chi$ by

$$\phi_0(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)k) = |a/d|_v^{1/2}\chi_1(a)\chi_2(d). \tag{18}$$

Then ϕ_0 spans the 1-dimensional space of K_v-fixed vectors in V_χ.

Proposition 4.4. *Let $q_v = \mathbb{N}(\mathfrak{p}_v)$. Then $\pi(f_{\mathfrak{p}_v^\ell})\phi_0 = \lambda_{\mathfrak{p}_v^\ell}\phi_0$, where*

$$\lambda_{\mathfrak{p}_v^\ell} = q_v^{\ell/2}\sum_{j=0}^{\ell}\chi_1(\varpi_v)^j\,\chi_2(\varpi_v)^{\ell-j}.$$

Proof. Because $f_{\mathfrak{p}_v^\ell}$ is left K_v-invariant, $R(f_{\mathfrak{p}_v^\ell})\phi_0$ is again fixed by K_v, and hence $R(f_{\mathfrak{p}_v^\ell})\phi_0 = \lambda\phi_0$ for some $\lambda \in \mathbf{C}$. The action of π is the same as the action of R, so it suffices to compute $\lambda = R(f_{\mathfrak{p}_v^\ell})\phi_0(1)$. Using the well-known left coset decomposition ([KL2], Lemma 13.4)

$$M(\mathfrak{p}_v^\ell, \mathfrak{N}_v) = \bigcup_{j=0}^{\ell}\ \bigcup_{a\in\mathcal{O}_v/\mathfrak{p}_v^j}\begin{pmatrix}\varpi_v^j & a\\ & \varpi_v^{\ell-j}\end{pmatrix}K_v,$$

we see that

$$\begin{aligned}\lambda &= \int_{\overline{G}_v} f_{\mathfrak{p}_v^\ell}(g)\phi_0(g)dg = \int_{\overline{M(\mathfrak{p}_v^\ell,\mathfrak{N}_v)}}\phi_0(g)dg\\ &= \sum_{j=0}^{\ell}\sum_{a\in\mathcal{O}_v/\mathfrak{p}_v^j}\phi_0\left(\begin{pmatrix}\varpi_v^j & a\\ & \varpi_v^{\ell-j}\end{pmatrix}\right) \qquad (\mathrm{meas}(\overline{K}_v)=1)\\ &= \sum_{j=0}^{\ell} q_v^j\left|\varpi_v^j/\varpi_v^{\ell-j}\right|_v^{1/2}\chi_1(\varpi_v)^j\chi_2(\varpi_v)^{\ell-j} = q_v^{\ell/2}\sum_{j=0}^{\ell}\chi_1(\varpi_v)^j\,\chi_2(\varpi_v)^{\ell-j},\end{aligned}$$

as claimed. ☐

Proposition 4.5. *With notation as above,*

$$q_v^{-\ell/2}\omega_v(\varpi_v)^{-\ell/2}\lambda_{\mathfrak{p}_v^\ell} = X_\ell(q_v^{-1/2}\omega_v(\varpi_v)^{-1/2}\lambda_{\mathfrak{p}_v}) \tag{19}$$

where

$$X_\ell(2\cos\theta) = \frac{\sin(\ell+1)\theta}{\sin\theta} = e^{i\ell\theta} + e^{i(\ell-2)\theta} + \cdots + e^{-i\ell\theta}$$

is the Chebyshev polynomial of degree ℓ.

Proof. Let $\alpha_{\mathfrak{p}} = \omega_v(\varpi_v)^{-1/2}\chi_1(\varpi_v)$ and $\beta_{\mathfrak{p}} = \omega_v(\varpi_v)^{-1/2}\chi_2(\varpi_v)$. Note that $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = 1$. Hence we may write $\alpha_{\mathfrak{p}} = e^{i\theta}$, $\beta_{\mathfrak{p}} = e^{-i\theta}$, and $\alpha_{\mathfrak{p}} + \beta_{\mathfrak{p}} = 2\cos\theta$ for some $\theta \in \mathbf{C}$. By the previous proposition, the left-hand side of (19) is

$$\sum_{j=0}^{\ell}\alpha_{\mathfrak{p}}^j\beta_{\mathfrak{p}}^{\ell-j} = \sum_{j=0}^{\ell}e^{ij\theta}e^{-i(\ell-j)\theta} = \sum_{j=0}^{\ell}e^{i(2j-\ell)\theta} = X_\ell(2\cos\theta).$$

This proves the result since $q_v^{-1/2}\omega(\varpi_v)^{-1/2}\lambda_{\mathfrak{p}_v} = \alpha_{\mathfrak{p}} + \beta_{\mathfrak{p}} = 2\cos\theta$. ☐

4.3. **Global Hecke operators.** Finally we define the global Hecke operator. Fix an ideal $\mathfrak{n}$ in $\mathcal{O}$, relatively prime to $\mathfrak{N}$. Define a function on $\mathbf{A}_{\mathrm{fin}}$ by

$$f_{\mathfrak{n}} = \prod_v f_{\mathfrak{n}_v},$$

where $f_{\mathfrak{n}_v}$ is defined as in the previous subsection. Then $f_{\mathfrak{n}}$ is bi-$K_1(\mathfrak{N})$-invariant, and supported on $Z(\mathbf{A}_{\mathrm{fin}})M(\mathfrak{n},\mathfrak{N})$, where

$$M(\mathfrak{n},\mathfrak{N}) = \prod_{v<\infty}M(\mathfrak{n}_v,\mathfrak{N}_v) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(\widehat{\mathcal{O}})\,|\, c\in\widehat{\mathfrak{N}},\ (ad-bc)\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}\right\}.$$

It is also clear that

$$f_{\mathfrak{n}}(zg) = \omega_{\mathrm{fin}}(z)^{-1} f_{\mathfrak{n}}(g) \quad (z \in Z(\mathbf{A}_{\mathrm{fin}}),\, g \in G(\mathbf{A}_{\mathrm{fin}})).$$

We define the operator

$$T_{\mathfrak{n}} = R(f_\infty \times f_{\mathfrak{n}})$$

on $L^2_0(\omega)$, which we can view as an operator on $A_{\mathbf{k}}(\mathfrak{N}, \omega)$ by Proposition 4.1. The family of operators $T_{\mathfrak{n}}$ for $(\mathfrak{n}, \mathfrak{N}) = 1$ is simultaneously diagonalizable (see Lemma 6.3 below).

The following proposition and its corollaries spell out the connection between Hecke eigenvalues and Fourier coefficients.

Proposition 4.6. *Given* $\mathfrak{n}$*, choose* $\mathtt{n}, \mathtt{d} \in \mathbf{A}^*_{\mathrm{fin}}$ *such that* $\mathtt{n}\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}$ *and* $\mathtt{d}\widehat{\mathcal{O}} = \widehat{\mathfrak{d}}$*. Then for any* $\varphi \in A_{\mathbf{k}}(\mathfrak{N}, \omega)$*,*

$$W_1^{T_{\mathfrak{n}}\varphi}(1/\mathtt{d}) = \mathbb{N}(\mathfrak{n}) W_1^{\varphi}(\mathtt{n}/\mathtt{d}).$$

Proof. We use the left coset decomposition

$$M(\mathfrak{n}, \mathfrak{N}) = \bigcup_{\substack{\mathtt{r},\mathtt{s} \in \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^* \\ \mathtt{rs}\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}}} \ \bigcup_{\mathtt{t} \in \widehat{\mathcal{O}}/\mathtt{r}\widehat{\mathcal{O}}} \begin{pmatrix} \mathtt{r} & \mathtt{t} \\ 0 & \mathtt{s} \end{pmatrix} K_0(\mathfrak{N}),$$

which is proven as in [KL2], Lemma 13.5. We see that

$$T_{\mathfrak{n}}\varphi(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}) = \int_{\overline{G}(\mathbf{A}_{\mathrm{fin}})} f_{\mathfrak{n}}(g)\varphi(\begin{pmatrix} y & x \\ & 1 \end{pmatrix} g)dg$$
$$= \sum_{\mathtt{r},\mathtt{s}} \sum_{\mathtt{t}} \varphi(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\begin{pmatrix} \mathtt{r} & \mathtt{t} \\ & \mathtt{s} \end{pmatrix}).$$

Note that

$$\varphi(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\begin{pmatrix} \mathtt{r} & \mathtt{t} \\ & \mathtt{s} \end{pmatrix}) = \varphi(\begin{pmatrix} y\mathtt{r} & y\mathtt{t} + x\mathtt{s} \\ & \mathtt{s} \end{pmatrix}) = \omega_{\mathrm{fin}}(\mathtt{s})\varphi(\begin{pmatrix} \frac{y\mathtt{r}}{\mathtt{s}} & \frac{y\mathtt{t}}{\mathtt{s}} + x \\ & 1 \end{pmatrix})$$
$$= \frac{\omega_{\mathrm{fin}}(\mathtt{s})}{d_F^{1/2}}\left[W_0^{\varphi}(\frac{y\mathtt{r}}{\mathtt{s}}) + \sum_{m \in F^*} W_1^{\varphi}(\frac{my\mathtt{r}}{\mathtt{s}})\theta_m(\frac{y\mathtt{t}}{\mathtt{s}} + x)\right].$$

Therefore

$$W_1^{T_{\mathfrak{n}}\varphi}(y) = d_F^{1/2} \cdot [\text{coeff. of } \theta_1(x)] = \sum_{\mathtt{r},\mathtt{s}} \omega_{\mathrm{fin}}(\mathtt{s}) W_1^{\varphi}(\frac{y\mathtt{r}}{\mathtt{s}}) \sum_{\mathtt{t}} \theta(-\frac{y\mathtt{t}}{\mathtt{s}}).$$

Now suppose $y_\infty = 1$ and identify y with y_{fin}. Then we can assume that $y\mathtt{r}/\mathtt{s} \in \widehat{\mathfrak{d}}^{-1}$ since otherwise $W_1^{\varphi}(\frac{y\mathtt{r}}{\mathtt{s}}) = 0$. Then $\theta(-y\mathtt{t}/\mathtt{s})$ is a well-defined character of $\mathtt{t} \in \widehat{\mathcal{O}}/\mathtt{r}\widehat{\mathcal{O}}$, and

$$\sum_{\mathtt{t} \in \widehat{\mathcal{O}}/\mathtt{r}\widehat{\mathcal{O}}} \theta(-y\mathtt{t}/\mathtt{s}) = \begin{cases} \mathbb{N}(\mathtt{r}) & \text{if } -y/\mathtt{s} \in \widehat{\mathfrak{d}}^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$W_1^{T_{\mathfrak{n}}(\varphi)}(y) = \sum_{\substack{\mathtt{r},\mathtt{s} \\ y/\mathtt{s} \in \widehat{\mathfrak{d}}^{-1}}} \omega_{\mathrm{fin}}(\mathtt{s})\mathbb{N}(\mathtt{r})W_1^{\varphi}(\frac{y\mathtt{r}}{\mathtt{s}}) \qquad (y \in \mathbf{A}^*_{\mathrm{fin}}). \tag{20}$$

Now take $y = 1/\mathtt{d}$. Then $y/\mathtt{s} \in \widehat{\mathfrak{d}}^{-1}$ only if $\mathtt{s} = 1 \in \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$. We can therefore take $\mathtt{r} = \mathtt{n}$ and $\mathtt{s} = 1$, so

$$W_1^{T_\mathfrak{n}\varphi}(1/\mathtt{d}) = \mathbb{N}(\mathfrak{n}) W_1^\varphi(\mathtt{n}/\mathtt{d}),$$

as claimed. □

Using the above, we can express Hecke eigenvalues in terms of Fourier coefficients and vice versa.

Corollary 4.7. *Suppose φ is an eigenvector of $T_\mathfrak{n}$ with eigenvalue $\lambda_\mathfrak{n}$. Then if $W_1^\varphi(1/\mathtt{d}) \neq 0$,*

$$\lambda_\mathfrak{n} = \frac{\mathbb{N}(\mathfrak{n}) W_1^\varphi(\mathtt{n}/\mathtt{d})}{W_1^\varphi(1/\mathtt{d})}.$$

Corollary 4.8. *If $(m\mathfrak{d}, \mathfrak{N}) = 1$, then for any $T_{m\mathfrak{d}}$-eigenfunction $\varphi \in A_\mathtt{k}(\mathfrak{N}, \omega)$ with $W_1^\varphi(1/\mathtt{d}) = 1$ and $T_{m\mathfrak{d}}\varphi = \lambda_{m\mathfrak{d}}\varphi$, we have*

$$W_m^\varphi(1) = \frac{e^{2\pi r}\prod_{j=1}^r \sigma_j(m)^{\mathtt{k}_j/2-1}}{d_F e^{2\pi \operatorname{tr}(m)}} \lambda_{m\mathfrak{d}}.$$

Proof. Apply Cor. 4.7 with $\mathfrak{n} = m\mathfrak{d}$ and $\mathtt{n} = m\mathtt{d}$. We get

$$\lambda_{m\mathfrak{d}} = \mathbb{N}(m\mathfrak{d}) W_1^\varphi(1_\infty \times m_{\text{fin}}) = \mathbb{N}(m\mathfrak{d}) W_m^\varphi(m_\infty^{-1} \times 1_{\text{fin}}). \tag{21}$$

Here $m_\infty = (\sigma_1(m), \dots, \sigma_r(m)) \in F_\infty^+$. Using (14), it is straightforward to show that

$$W_m^\varphi(m_\infty^{-1}) = \left(\prod_{j=1}^r \sigma_j(m)^{-\mathtt{k}_j/2}\right) e^{2\pi \operatorname{tr}(m)} e^{-2\pi \operatorname{tr}(1)} W_m^\varphi(1).$$

Substituting this into (21) and using $\mathbb{N}(m\mathfrak{d}) = d_F \mathbb{N}(m)$ gives the result. □

5. A Hilbert modular Petersson trace formula

Let $f = f_\infty \times f_\mathfrak{n}$ for an ideal $(\mathfrak{n}, \mathfrak{N}) = 1$, and recall that $T_\mathfrak{n} = R(f)$ is the associated Hecke operator. Let $\mathcal{F}$ be any orthogonal basis for $A_\mathtt{k}(\mathfrak{N}, \omega)$. (Later we will require $\mathcal{F}$ to consist of eigenvectors of $T_\mathfrak{n}$.) Then the kernel of $R(f)$ is given by

$$K(x, y) = \sum_{\varphi \in \mathcal{F}} \frac{R(f)\varphi(x)\overline{\varphi(y)}}{\|\varphi\|^2} = \sum_{\gamma \in \overline{G}(F)} f(x^{-1}\gamma y),$$

The first expression is the spectral expansion of the kernel, and the second is the geometric expansion. The equality of the two hinges on the continuity (in x, y) of the geometric expansion. The proof of this continuity given in Prop 3.2. of [Li2] carries over easily to the case of nontrivial central character we consider here.

In this section we apply the technique of Section 2 to the kernel function given above, taking $H = N \times N$. We need to fix a character on $N(F)\backslash N(\mathbf{A}) \times N(F)\backslash N(\mathbf{A})$, which amounts to choosing two characters on $F\backslash \mathbf{A}$. As discussed earlier, every character on $F\backslash \mathbf{A}$ is of the form

$$\theta_m(x) = \theta(-mx)$$

for some $m \in F$. Fix $m_1, m_2 \in F$. Our goal is to obtain a trace formula by computing

$$\int_{N(F)\backslash N(\mathbf{A})} \int_{N(F)\backslash N(\mathbf{A})} K(n_1, n_2) \overline{\theta_{m_1}(n_1)} \theta_{m_2}(n_2) dn_1 dn_2 \tag{22}$$

with the two expressions for the kernel.

5.1. **The spectral side.** Using the spectral expansion of the kernel, expression (22) is easily computed in terms of Hecke eigenvalues and Fourier coefficients of cusp forms. Suppose the basis $\mathcal{F}$ consists of eigenfunctions of $T_{\mathfrak{n}}$. Then for $\varphi \in \mathcal{F}$ we have $R(f)\varphi = \lambda_{\mathfrak{n}}^{\varphi}\varphi$ for some scalar $\lambda_{\mathfrak{n}}^{\varphi} \in \mathbf{C}$. Hence (22) is

$$= \sum_{\varphi\in\mathcal{F}} \frac{\lambda_{\mathfrak{n}}^{\varphi}}{\|\varphi\|^2} \int_{N(F)\backslash N(\mathbf{A})} \varphi(n_1)\overline{\theta_{m_1}(n_1)}dn_1 \int_{N(F)\backslash N(\mathbf{A})} \overline{\varphi(n_2)}\theta_{m_2}(n_2)dn_2$$

$$= \sum_{\varphi\in\mathcal{F}} \frac{\lambda_{\mathfrak{n}}^{\varphi} W_{m_1}^{\varphi}(1)\overline{W_{m_2}^{\varphi}(1)}}{\|\varphi\|^2}, \tag{23}$$

as in (10). By Proposition 3.3, the above expression is nonzero only if

$$m_1, m_2 \in \mathfrak{d}_+^{-1}.$$

We may assume that this holds from now on.

5.2. **The geometric side.** Here we use the method of Section 2 to compute (22) using the geometric expansion of the kernel. This gives a sum $\sum_{[\delta]} I_\delta(f)$, where

$$I_\delta(f) = \int_{H_\delta(F)\backslash H(\mathbf{A})} f(n_1^{-1}\delta n_2)\overline{\theta_{m_1}(n_1)}\theta_{m_2}(n_2)dn_1dn_2.$$

The orbits $[\delta]$ are in one-to-one correspondence with the double cosets

$$N(F)\backslash\overline{G}(F)/N(F).$$

Let M be the group of invertible diagonal matrices. The Bruhat decomposition is the following partition of $G(F)$ into two cells:

$$G(F) = N(F)M(F) \cup N(F)M(F)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}N(F).$$

We call these the first and second **Bruhat cells** respectively. This gives

$$N(F)\backslash\overline{G}(F)/N(F) = \{\,[(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix})]|\ \gamma\in F^*\}\bigcup\{\,[(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix})]|\ \mu\in F^*\}.$$

We need to determine which of these orbits are relevant in the sense of §2.

First let $\delta = (\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}) \in G(F)$. If $((\begin{smallmatrix}1 & t_1\\ & 1\end{smallmatrix}), (\begin{smallmatrix}1 & t_2\\ & 1\end{smallmatrix})) \in H_\delta(\mathbf{A})$, then

$$\begin{pmatrix}1 & -t_1\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\\ & 1\end{pmatrix} = z\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix},$$

for some $z \in Z(F)$. A simple calculation shows that $z = 1$ and $t_1 = \gamma t_2$, so

$$H_\delta(\mathbf{A}) = \left\{(\begin{pmatrix}1 & \gamma t\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix})|\, t\in\mathbf{A}\right\}. \tag{24}$$

Thus δ is relevant if and only if

$$\theta((m_1\gamma - m_2)t) = 1$$

for all $t \in \mathbf{A}$, or equivalently, $\gamma = m_2/m_1$ (since $m_1 \in \mathfrak{d}_+^{-1}$ is nonzero).

On the other hand, if $\delta = \begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix} \in G(F)$, one sees easily that

$$H_\delta(\mathbf{A}) = \{(e,e)\},$$

so all of these matrices are relevant.

5.2.1. *Computation of the first type of I_δ.* Here we take $m_1, m_2 \in \mathfrak{d}_+^{-1}$, and $\delta = \left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)$ where $\gamma = m_2/m_1$. By (24),

$$I_\delta(f) = \int_{\{(\gamma t,t)\in F^2\}\backslash \mathbf{A}\times\mathbf{A}} f(\begin{pmatrix}1 & -t_1\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\\ & 1\end{pmatrix})\theta(m_1t_1 - m_2t_2)dt_1dt_2$$
$$= \int_{\{(\gamma t,t)\in F^2\}\backslash \mathbf{A}\times\mathbf{A}} f(\begin{pmatrix}\gamma & \gamma t_2 - t_1\\ 0 & 1\end{pmatrix})\theta(m_1t_1 - m_2t_2)dt_1dt_2.$$

Let $t_1' = \gamma t_2 - t_1$ and $t_2' = t_2$. Then because $m_1\gamma = m_2$, we have $m_1t_1 - m_2t_2 = -m_1t_1'$, so

$$I_\delta = \int_{\{0\}\times F\backslash(\mathbf{A}\times\mathbf{A})} f(\begin{pmatrix}\gamma & t_1'\\ 0 & 1\end{pmatrix})\theta(-m_1t_1')dt_1'dt_2'$$
$$= \text{meas}(F\backslash\mathbf{A})\int_{\mathbf{A}} f(\begin{pmatrix}m_2/m_1 & t\\ 0 & 1\end{pmatrix})\theta(-m_1t)dt$$
$$= d_F^{1/2}\int_{\mathbf{A}} f(\begin{pmatrix}m_2 & m_1t\\ & m_1\end{pmatrix})\theta(-m_1t)dt$$
$$= d_F^{1/2}\int_{\mathbf{A}} f(\begin{pmatrix}m_2 & t\\ 0 & m_1\end{pmatrix})\theta(-t)dt.$$

Here we used (6) and the fact that $f(zg) = f(g)$ for $z \in Z(F)$.

We factorize the above integral into $(I_\delta)_\infty(I_\delta)_{\text{fin}}$, and incorporate the coefficient $d_F^{1/2}$ into $(I_\delta)_\infty$. First we consider

$$(I_\delta)_{\text{fin}} = \int_{\mathbf{A}_{\text{fin}}} f_{\mathfrak{n}}(\begin{pmatrix}m_2 & t\\ & m_1\end{pmatrix})\theta_{\text{fin}}(-t)dt.$$

We shall compute this locally, as we may since $f_{\mathfrak{n}_v}(\left(\begin{smallmatrix}m_2 & t\\ & m_1\end{smallmatrix}\right))\theta_v(-t)\big|_{t\in\mathcal{O}_v} \equiv 1$ for almost all v. For any finite place v we have

$$(I_\delta)_v = \int_{F_v} f_{\mathfrak{n}_v}(\begin{pmatrix}m_2 & t\\ & m_1\end{pmatrix})\theta_v(-t)dt.$$

The integrand is nonzero only if

$$\text{ord}_v(\frac{m_1m_2}{\mathbf{s}_v^2}) = \text{ord}_v(\mathfrak{n}_v)$$

for some $\mathbf{s}_v \in F_v^*$. Supposing this is the case,

$$(I_\delta)_v = \int_{F_v} f_{\mathfrak{n}_v}(\begin{pmatrix}\mathbf{s}_v & \\ & \mathbf{s}_v\end{pmatrix}\begin{pmatrix}\frac{m_2}{\mathbf{s}_v} & \frac{t}{\mathbf{s}_v}\\ & \frac{m_1}{\mathbf{s}_v}\end{pmatrix})\theta_v(-t)dt$$

is nonzero only if $\frac{m_2}{\mathbf{s}_v}, \frac{m_1}{\mathbf{s}_v}, \frac{t}{\mathbf{s}_v} \in \mathcal{O}_v$ by Lemma 4.2. Assuming this, we have

$$(I_\delta)_v = \int_{\mathbf{s}_v\mathcal{O}_v} \omega_v(\mathbf{s}_v)^{-1} f_{\mathfrak{n}_v}(\begin{pmatrix}\frac{m_2}{\mathbf{s}_v} & \frac{t}{\mathbf{s}_v}\\ & \frac{m_1}{\mathbf{s}_v}\end{pmatrix})\theta_v(-t)dt.$$

Suppose $v|\mathfrak{N}$. Then by the definition of $f_{\mathfrak{n}_v}$ (see (17)),

$$(I_\delta)_v = \omega_v(\mathbf{s}_v)^{-1}\omega_v(\frac{m_1}{\mathbf{s}_v})^{-1}\psi(\mathfrak{N}_v)\int_{\mathbf{s}_v\mathcal{O}_v}\theta_v(-t)dt.$$

The integral vanishes unless $\mathbf{s}_v \in \mathfrak{d}_v^{-1}$, in which case its value is $|\mathbf{s}_v|_v$. Hence

$$(I_\delta)_v = |\mathbf{s}_v|_v\,\omega_v(\mathbf{s}_v)^{-1}\omega_v(\frac{m_1}{\mathbf{s}_v})^{-1}\psi(\mathfrak{N}_v), \quad v|\mathfrak{N}.$$

Now suppose $v \nmid \mathfrak{N}$. Then

$$(I_\delta)_v = \omega_v(\mathtt{s}_v)^{-1} \int_{\mathtt{s}_v \mathcal{O}_v} \theta_v(-t)dt,$$

which as before vanishes unless $\mathtt{s}_v \in \mathfrak{d}_v^{-1}$. So in this case,

$$(I_\delta)_v = |\mathtt{s}_v|_v \, \omega_v(\mathtt{s}_v)^{-1}, \quad v \nmid \mathfrak{N}.$$

Define

$$\omega_{\mathfrak{N},v} = \begin{cases} \omega_v & \text{if } v|\mathfrak{N} \\ 1 & \text{if } v \nmid \mathfrak{N}, \end{cases} \tag{25}$$

and let $\omega_{\mathfrak{N}} = \prod \omega_{\mathfrak{N},v} = \prod_{v|\mathfrak{N}} \omega_v$. Notice that $\omega_{\mathfrak{N}}$ is a character on $\mathbf{A}_{\text{fin}}^*$. Multiplying the above local results together, we obtain the following.

Proposition 5.1. *Suppose* $\delta = \begin{pmatrix} \frac{m_2}{m_1} & 0 \\ 0 & 1 \end{pmatrix} \in G(F)$. *The integral* $(I_\delta)_{\text{fin}}$ *is nonzero if and only if there exists a finite idele* $\mathtt{s} \in \mathbf{A}_{\text{fin}}^*$ *such that*

- $\frac{m_1}{\mathtt{s}}, \frac{m_2}{\mathtt{s}} \in \widehat{\mathcal{O}}$
- $\text{ord}_v(\frac{m_1 m_2}{\mathtt{s}_v^2}) = \text{ord}_v(\mathfrak{n})$ *for all* $v < \infty$
- $\mathtt{s} \in \widehat{\mathfrak{d}}^{-1}$.

Under these conditions,

$$(I_\delta)_{\text{fin}} = |\mathtt{s}|_{\text{fin}} \omega_{\text{fin}}(\mathtt{s})^{-1} \psi(\mathfrak{N}) \omega_{\mathfrak{N}}(m_1/\mathtt{s})^{-1}$$

where $\psi(\mathfrak{N}) = [K_{\text{fin}} : K_0(\mathfrak{N})]$.

Remark: The expression is independent of the choice of $\mathtt{s}$. Indeed if $\mathtt{s}' = u\mathtt{s}$ for $u \in \widehat{\mathcal{O}}^*$, then

$$\omega_{\text{fin}}(\mathtt{s}')^{-1} \omega_{\mathfrak{N}}(\mathtt{s}') = \prod_{v \nmid \mathfrak{N}} \omega_v(\mathtt{s}'_v) = \prod_{v \nmid \mathfrak{N}} \omega_v(\mathtt{s}_v) = \omega_{\text{fin}}(\mathtt{s})^{-1} \omega_{\mathfrak{N}}(\mathtt{s})$$

since ω_v is unramified for $v \nmid \mathfrak{N}$.

For purposes of computation, the following lemma is helpful and easily verified.

Lemma 5.2. *With notation as above,* $(I_\delta)_{\text{fin}}$ *is nonzero if and only if for every finite valuation* v *of* F,

- $\xi_v \overset{\text{def}}{=} \frac{1}{2} \text{ord}_v(m_1 m_2 \mathfrak{n}^{-1}) \in \mathbf{Z}$
- $\text{ord}_v(m_i) \geq \xi_v \geq -\text{ord}_v(\mathfrak{d}_v)$ *for* $i = 1, 2$.

If these conditions hold, then we can define $\mathtt{s}$ *by taking* $\mathtt{s}_v = \varpi_v^{\xi_v}$ *for all* v.

For the infinite part, we have the following.

Proposition 5.3. *Let* $\delta = \begin{pmatrix} \frac{m_2}{m_1} & 0 \\ 0 & 1 \end{pmatrix}$ *with* $m_1, m_2 \in F^*$. *Then* $(I_\delta)_\infty$ *is nonzero if and only if* $m_1, m_2 \in F^+$. *Under this condition,*

$$(I_\delta)_\infty = \frac{d_F^{1/2}}{e^{2\pi \, \text{tr}_{\mathbf{Q}}^F (m_1 + m_2)}} \prod_{j=1}^r \frac{(4\pi)^{\mathtt{k}_j - 1} \sigma_j(m_1 m_2)^{\mathtt{k}_j/2}}{(\mathtt{k}_j - 2)!}.$$

Proof. We have

$$(I_\delta)_\infty = d_F^{1/2}\prod_{j=1}^{r}\int_{-\infty}^{\infty} f_{\infty_j}(\begin{pmatrix}\sigma_j(m_2) & t\\ & \sigma_j(m_1)\end{pmatrix})\theta_{\infty_j}(-t)dt.$$

The integrand is nonzero only if $\sigma_j(m_1m_2)>0$. By the formula for f_{∞_j},

$$\int_{-\infty}^{\infty} f_{\infty_j}(\begin{pmatrix}\sigma_j(m_2) & t\\ & \sigma_j(m_1)\end{pmatrix})\theta_{\infty_j}(-t)dt$$

$$=\frac{\mathtt{k}_j-1}{4\pi}\sigma_j(m_1m_2)^{\mathtt{k}_j/2}(2i)^{\mathtt{k}_j}\int_{-\infty}^{\infty}\frac{e^{2\pi i t}}{(-t+i\sigma_j(m_1+m_2))^{\mathtt{k}_j}}\,dt.$$

Using a complex contour integral, this is nonzero only if $\sigma_j(m_1+m_2)>0$, in which case it equals

$$\frac{(4\pi)^{\mathtt{k}_j-1}}{(\mathtt{k}_j-2)!}\sigma_j(m_1m_2)^{\mathtt{k}_j/2}e^{-2\pi\sigma_j(m_1+m_2)}.$$

Full details are given in [KL1], Proposition 3.4. Because $\sigma_j(m_1)$ and $\sigma_j(m_2)$ have the same sign, the condition $\sigma_j(m_1+m_2)>0$ is equivalent to $\sigma_j(m_1),\sigma_j(m_2)>0$. □

Proposition 5.4. *Let* $\delta=\begin{pmatrix}\frac{m_2}{m_1} & 0\\ 0 & 1\end{pmatrix}$ *for* $m_1,m_2\in\mathfrak{d}_+^{-1}$. *Then* $I_\delta(f)$ *is nonzero if and only if* $\frac{m_1m_2}{\mathtt{s}^2}\widehat{\mathcal{O}}=\widehat{\mathfrak{n}}$ *and* $\frac{m_1}{\mathtt{s}},\frac{m_2}{\mathtt{s}}\in\widehat{\mathcal{O}}$ *for some* $\mathtt{s}\in\widehat{\mathfrak{d}}^{-1}$. *Under this condition,*

$$I_\delta=\left[\prod_{j=1}^{r}\frac{\left(4\pi\sqrt{\sigma_j(m_1m_2)}\right)^{\mathtt{k}_j-1}}{(\mathtt{k}_j-2)!}\right]\frac{d_F^{1/2}\mathbb{N}(\mathfrak{n})^{1/2}\psi(\mathfrak{N})}{e^{2\pi\,\mathrm{tr}_{\mathbf{Q}}^{F}(m_1+m_2)}\omega_{\mathfrak{N}}(m_1/\mathtt{s})\omega_{\mathrm{fin}}(\mathtt{s})}.$$

Proof. By Lemma 5.2, we can take $\mathtt{s}_v=\varpi_v^{(\mathrm{ord}_v(m_1m_2)-\mathrm{ord}_v(\mathfrak{n}))/2}$. Choose $\mathtt{n}\in\mathbf{A}_{\mathrm{fin}}^*$ such that $\mathtt{n}\widehat{\mathcal{O}}=\widehat{\mathfrak{n}}$. Then

$$|\mathtt{s}|_{\mathrm{fin}}=\left(\frac{|m_1m_2|_{\mathrm{fin}}}{|\mathtt{n}|_{\mathrm{fin}}}\right)^{1/2}.$$

Note that by the product formula and the fact that $m_1,m_2\in F^+$,

$$|m_1m_2|_{\mathrm{fin}}=\prod_{j=1}^{r}\sigma_j(m_1m_2)^{-1}.$$

Hence

$$|\mathtt{s}|_{\mathrm{fin}}=\mathbb{N}(\mathfrak{n})^{1/2}\prod_{j=1}^{r}\sigma_j(m_1m_2)^{-1/2}.$$

The proposition now follows immediately upon multiplying $(I_\delta)_{\mathrm{fin}}$ and $(I_\delta)_\infty$ together. □

5.2.2. *Computation of the second type of I_δ.* Let $v < \infty$ be a finite valuation of F. Let $\mathtt{n}_v \in \mathcal{O}_v - \{0\}$ and let $\mathtt{m}_{1v}, \mathtt{m}_{2v} \in \mathfrak{d}_v^{-1}$. Recall that the conductor of ω_v divides $\mathfrak{N}_v$. For $\mathtt{c}_v \in \mathfrak{N}_v - \{0\}$, define a generalized (local) Kloosterman sum by

$$S_{\omega_v}(\mathtt{m}_{1v}, \mathtt{m}_{2v}; \mathtt{n}_v; \mathtt{c}_v) = \sum_{\substack{s_1, s_2 \in \mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v \\ s_1 s_2 \equiv \mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v}} \theta_v(\frac{\mathtt{m}_{1v}s_1 + \mathtt{m}_{2v}s_2}{\mathtt{c}_v})\omega_v(s_2)^{-1}.$$

The value of the sum is 1 if $\mathtt{c}_v \in \mathcal{O}_v^*$. (The above notation is not quite consistent with [KL1], which has $\omega_v(s_1)^{-1}$.)

For $\mathtt{n} \in \widehat{\mathcal{O}} \cap \mathbf{A}_{\text{fin}}^*$, $\mathtt{c} \in \widehat{\mathfrak{N}} \cap \mathbf{A}_{\text{fin}}^*$, $\mathtt{m}_1, \mathtt{m}_2 \in \widehat{\mathfrak{d}}^{-1}$, and $\omega_{\mathfrak{N}}$ as in (25), we define

$$S_{\omega_{\mathfrak{N}}}(\mathtt{m}_1, \mathtt{m}_2; \mathtt{n}; \mathtt{c}) = \sum_{\substack{s_1, s_2 \in \widehat{\mathcal{O}}/\mathtt{c}\widehat{\mathcal{O}} \\ s_1 s_2 \equiv \mathtt{n} \bmod \mathtt{c}\widehat{\mathcal{O}}}} \theta_{\text{fin}}(\frac{\mathtt{m}_1 s_1 + \mathtt{m}_2 s_2}{\mathtt{c}})\omega_{\mathfrak{N}}(s_2)^{-1}.$$

Then

$$S_{\omega_{\mathfrak{N}}}(\mathtt{m}_1, \mathtt{m}_2; \mathtt{n}; \mathtt{c}) = \prod_{v<\infty} S_{\omega_{\mathfrak{N},v}}(\mathtt{m}_{1v}, \mathtt{m}_{2v}; \mathtt{n}_v; \mathtt{c}_v).$$

When $\delta = \left(\begin{smallmatrix} 0 & \mu \\ 1 & 0 \end{smallmatrix}\right)$, we have seen that $H_\delta = \{(e,e)\}$, so

$$I_\delta(f) = \iint_{N(\mathbf{A})\times N(\mathbf{A})} f(n_1^{-1}\begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix} n_2)\,\overline{\theta_{m_1}(n_1)}\,\theta_{m_2}(n_2)dn_1dn_2.$$

We will compute this locally. Write $n_i = \left(\begin{smallmatrix} 1 & t_i \\ & 1 \end{smallmatrix}\right)$, $i = 1, 2$. Then

$$n_1^{-1}\begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix} n_2 = \begin{pmatrix} -t_1 & \mu - t_1t_2 \\ 1 & t_2 \end{pmatrix}. \tag{26}$$

Proposition 5.5. *Let $m_1, m_2 \in \mathfrak{d}^{-1}$. If $\delta = \begin{pmatrix} & \mu \\ 1 & \end{pmatrix}$, then $(I_\delta)_{\text{fin}}$ is nonzero only if $\mathtt{c}^2\mu\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}$ for some $\mathtt{c} \in \widehat{\mathfrak{N}}$. Under this condition, let $\mathtt{n} = -\mathtt{c}^2\mu$ be a generator of $\widehat{\mathfrak{n}}$ (the negative sign is for convenience). Then*

$$(I_\delta)_{\text{fin}} = \left[\prod_{j=1}^{r}(-1)^{\mathtt{k}_j}\right]\psi(\mathfrak{N})\omega_{\text{fin}}(\mathtt{c})S_{\omega_{\mathfrak{N}}}(m_1, m_2; \mathtt{n}; \mathtt{c}).$$

Proof. For a discrete place v,

$$(I_\delta)_v = \iint_{F_v\times F_v} f_{\mathfrak{n}_v}(\begin{pmatrix} -t_1 & \mu - t_1t_2 \\ 1 & t_2 \end{pmatrix})\theta_v(m_1t_1 - m_2t_2)dt_1dt_2.$$

The integrand is nonzero only if there exists $\mathtt{c}_v \in F_v^*$ such that

$$\begin{pmatrix} \mathtt{c}_v & \\ & \mathtt{c}_v \end{pmatrix}\begin{pmatrix} -t_1 & \mu - t_1t_2 \\ 1 & t_2 \end{pmatrix} \in M(\mathfrak{n}_v, \mathfrak{N}_v).$$

This means

(1) $\mathtt{c}_v \in \mathfrak{N}_v$
(2) $\mathtt{c}_v t_1, \mathtt{c}_v t_2 \in \mathcal{O}_v$
(3) $\text{ord}_v(\mathtt{c}_v^2\mu) = \text{ord}_v(\mathfrak{n}_v)$.

Note that the first and third conditions determine $\mathtt{c}_v$ up to units. By the third condition, $\mathtt{c}_v^2\mu = -\mathtt{n}_v$ for some generator $\mathtt{n}_v \in \mathfrak{n}_v$. Now make substitutions in the integral by replacing t_1 and t_2 by $\frac{1}{\mathtt{c}_v}t_1$ and $\frac{1}{\mathtt{c}_v}t_2$ respectively. Then

$$(I_\delta)_v = |\mathtt{c}_v|_v^{-2}\iint_{\mathcal{O}_v\times\mathcal{O}_v} f_{\mathtt{n}_v}(\mathtt{c}_v^{-1}\begin{pmatrix}-t_1 & \frac{-\mathtt{n}_v - t_1t_2}{\mathtt{c}_v}\\ \mathtt{c}_v & t_2\end{pmatrix})\theta_v(\frac{m_1t_1-m_2t_2}{\mathtt{c}_v})dt_1dt_2.$$

The integrand is nonzero if and only if

(1) $\mathtt{c}_v \in \mathfrak{N}_v$
(2) $t_1t_2 \equiv -\mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v$.

Assuming these hold, the value

$$f_{\mathtt{n}_v}(\mathtt{c}_v^{-1}\begin{pmatrix}-t_1 & \frac{-\mathtt{n}_v - t_1t_2}{\mathtt{c}_v}\\ \mathtt{c}_v & t_2\end{pmatrix}) = \begin{cases}\omega_v(\mathtt{c}_v)\psi(\mathfrak{N}_v)\omega_v(t_2)^{-1} & \text{if } v|\mathfrak{N}\\ \omega_v(\mathtt{c}_v) & \text{otherwise}\end{cases}$$

depends only on the residue class of t_2 modulo $\mathfrak{N}_v$. Furthermore, because θ_v is trivial on $\mathfrak{d}_v^{-1}$ and $m_1, m_2 \in \mathfrak{d}^{-1}$, the value $\theta_v(\frac{m_1t_1-m_2t_2}{\mathtt{c}_v})$ depends only on the cosets $t_1 + \mathtt{c}_v\mathcal{O}_v$ and $t_2 + \mathtt{c}_v\mathcal{O}_v$. Thus the entire integrand is constant on cosets of $\mathtt{c}_v\mathcal{O}_v$. Each of these cosets has measure $|\mathtt{c}_v|_v$, so these measures for t_1 and t_2 will cancel the coefficient $|\mathtt{c}_v|_v^{-2}$ in the integral. Therefore

$$(I_\delta)_v = \begin{cases}\psi(\mathfrak{N}_v)\omega_v(\mathtt{c}_v)\displaystyle\sum_{\substack{s_1,s_2\in\mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v\\ s_1s_2\equiv-\mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v}}\omega_v(s_2)^{-1}\theta_v(\frac{m_1s_1-m_2s_2}{\mathtt{c}_v}) & \text{if } v|\mathfrak{N}\\ \omega_v(\mathtt{c}_v)\displaystyle\sum_{\substack{s_1,s_2\in\mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v\\ s_1s_2\equiv-\mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v}}\theta_v(\frac{m_1s_1-m_2s_2}{\mathtt{c}_v}) & \text{if } v\nmid\mathfrak{N}.\end{cases}$$

Replacing s_2 by $-s_2$, in either case we see that

$$(I_\delta)_v = \psi(\mathfrak{N}_v)\omega_{\mathfrak{N},v}(-1)\omega_v(\mathtt{c}_v)S_{\omega_{\mathfrak{N},v}}(m_1,m_2;\mathtt{n}_v;\mathtt{c}_v).$$

Multiplying the local results together, we obtain

$$(I_\delta)_{\text{fin}} = \psi(\mathfrak{N})\omega_{\mathfrak{N}}(-1)\omega_{\text{fin}}(\mathtt{c})S_{\omega_{\mathfrak{N}}}(m_1,m_2;\mathtt{n};\mathtt{c}).$$

The final point is that because $\omega(-1) = 1$ and ω_v is unramified for $v\nmid\mathfrak{N}$,

$$\omega_{\mathfrak{N}}(-1) = \prod_{v|\mathfrak{N}}\omega_v(-1) = \prod_{v<\infty}\omega_v(-1) = \omega_\infty(-1)^{-1} = \prod_{j=1}^{r}(-1)^{\mathtt{k}_j}.$$

□

Proposition 5.6. *Let* $\delta = \begin{pmatrix} & \mu\\ 1 & \end{pmatrix} \in G(F)$. *Then*

$$(I_\delta)_\infty = \iint_{F_\infty\times F_\infty} f_\infty(\begin{pmatrix}-t_1 & \mu - t_1t_2\\ 1 & t_2\end{pmatrix})\theta_\infty(m_1t_1-m_2t_2)dt_1dt_2 \tag{27}$$

is nonzero only if $m_1, m_2, -\mu$ *are all totally positive. Under these conditions,*

$$(I_\delta)_\infty = \frac{\mathbb{N}(-\mu)^{1/2}}{e^{2\pi\,\mathrm{tr}_{\mathbb{Q}}^F(m_1+m_2)}}\prod_{j=1}^{r}\frac{(4\pi i)^{\mathtt{k}_j}\sqrt{\sigma_j(m_1m_2)}^{\mathtt{k}_j-1}J_{\mathtt{k}_j-1}(4\pi\sqrt{-\sigma_j(\mu m_1m_2)})}{2(\mathtt{k}_j-2)!}$$

where J_k *is the Bessel* J*-function.*

Proof. The computation of $(I_\delta)_{\infty_j}$ is given in [KL1] Proposition 3.6, and the above is the product of these local computations. Note that since $-\mu \in F^+$, $\mathbb{N}(-\mu) = \mathrm{N}(-\mu) = \prod_{j=1}^r \sigma_j(-\mu)$. □

Multiplying the above results together, we obtain the following.

Proposition 5.7. *Let* $\delta = \begin{pmatrix} & \mu \\ 1 & \end{pmatrix} \in G(F)$, *and* $m_1, m_2 \in \mathfrak{d}_+^{-1}$. *Then* I_δ *is nonzero only if*

(1) $-\mu$ *is totally positive*
(2) $\mathtt{c}^2\mu\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}$ *for some* $\mathtt{c} \in \widehat{\mathfrak{N}} \cap \mathbf{A}_{\mathrm{fin}}^*$.

Under these conditions, letting $\mathtt{n} = -\mathtt{c}^2\mu$, *we have*

$$I_\delta = \psi(\mathfrak{N})\omega_{\mathrm{fin}}(\mathtt{c})S_{\omega_{\mathfrak{N}}}(m_1, m_2; \mathtt{n}; \mathtt{c})\frac{\mathbb{N}(-\mu)^{1/2}}{2^r e^{2\pi\,\mathrm{tr}_{\mathbf{Q}}^F(m_1+m_2)}}$$
$$\times \prod_{j=1}^r \frac{(-4\pi i)^{\mathtt{k}_j}\sqrt{\sigma_j(m_1m_2)}^{\mathtt{k}_j-1}}{(\mathtt{k}_j-2)!} J_{\mathtt{k}_j-1}(4\pi\sqrt{-\sigma_j(\mu m_1 m_2)}).$$

5.2.3. *Total contribution of the second Bruhat cell.* The total contribution of the second type of δ is a summation of I_δ over all $\mu \in F^*$ satisfying the conditions in Proposition 5.7. For this we need to give a more systematic description of the set of such μ.

Let $\mathtt{c} \in \mathbf{A}_{\mathrm{fin}}^*$, and let $\mathfrak{c} = \mathtt{c}\widehat{\mathcal{O}} \cap F$ be the associated fractional ideal of F. Then condition (2) of Proposition 5.7 is equivalent to

$$\mathfrak{c}^2(\mu) = \mathfrak{n} \tag{28}$$

and

$$\mathfrak{N}|\mathfrak{c}. \tag{29}$$

We need to determine the elements μ which satisfy (28) for some $\mathfrak{c}$ as in (29).

Consider the following equation in the ideal class group

$$1 = [\mathfrak{b}]^2[\mathfrak{n}]. \tag{30}$$

If no solution $\mathfrak{b}$ exists, then the contribution of this Bruhat cell is 0. Some examples are given at the end of this section. Otherwise, let

$$[\mathfrak{b}_1], [\mathfrak{b}_2], \ldots, [\mathfrak{b}_t]$$

be the distinct solutions of (30). We take the $\mathfrak{b}_i$ to be integral ideals. Because $\mathfrak{b}_i^2\mathfrak{n}$ is principal, there exists a nonzero element $\eta_i \in \mathcal{O}$ such that

$$(\eta_i) = \mathfrak{b}_i^2\mathfrak{n}. \tag{31}$$

We fix such generators $\eta_1, \ldots, \eta_t$ once and for all.

Suppose (28) holds. Then $[\mathfrak{c}]^{-1} = [\mathfrak{b}_i]$ for some i, so $\mathfrak{c} = s\mathfrak{b}_i^{-1}$ for some $s \in F^*$. Substituting $\mathfrak{c} = s\mathfrak{b}_i^{-1}$ into (28), we see that $(\mu) = (s^{-2}\eta_i)$, i.e. $\mu = \frac{-\eta_i u}{s^2}$ for some $u \in \mathcal{O}^*$. Conversely if $\mu = \frac{-\eta_i u}{s^2}$ for some $s \in F^*$, then it satisfies (28) with $\mathfrak{c} = s\mathfrak{b}_i^{-1}$.

For each $i = 1, \ldots, t$ fix a generator $\mathtt{b}_i \in \widehat{\mathfrak{b}}_i$. We obtain the following lemma:

Lemma 5.8. *An element $\mu \in F^*$ satisfies condition* (2) *of Proposition 5.7 if and only if*

$$\mu = \frac{-\eta_i u}{s^2} \tag{32}$$

for some $i \in \{1, \ldots, t\}$, $u \in \mathcal{O}^*$ and $s \in \mathfrak{b}_i \mathfrak{N}$. If this condition holds, then we can take $\mathtt{c} = s\mathtt{b}_i^{-1}$ and $\mathtt{n} = \eta_i u \mathtt{b}_i^{-2}$ in Proposition 5.7.

Proof. The above discussion shows that μ satisfies (28) if and only if it is given by (32) where $\mathfrak{c} = s\mathfrak{b}_i^{-1}$. Then it is easy to check that $\mathfrak{N}|\mathfrak{c}$ if and only if $s \in \mathfrak{b}_i\mathfrak{N}$. The last part of the lemma follows by a simple calculation. □

Lemma 5.9. *In the notation above, if*

$$\frac{-\eta_i u}{s^2} = \frac{-\eta_{i'} u'}{s'^2},$$

then $i = i'$ and $u\mathcal{O}^{*2} = u'\mathcal{O}^{*2}$. Furthermore if we assume $u, u' \in U$ as in (4)*, then $u = u'$ and $s = \pm s'$.*

Proof. By (31), we have $\mathfrak{b}_i^2 \mathfrak{n} s^{-2} = \mathfrak{b}_{i'}^2 \mathfrak{n} s'^{-2}$. Therefore $(\mathfrak{b}_i \mathfrak{b}_{i'}^{-1})^2 = (ss'^{-1})^2$. By unique factorization of ideals, we have $\mathfrak{b}_i \mathfrak{b}_{i'}^{-1} = (ss'^{-1})$. Therefore $[\mathfrak{b}_i] = [\mathfrak{b}_{i'}]$, and hence $i = i'$. With $i = i'$, we have $uu'^{-1} = (ss'^{-1})^2$. This implies that $uu'^{-1} \in \mathcal{O}^* \cap F^{*2} = \mathcal{O}^{*2}$. Then if $u, u' \in U$, it is immediate that $u = u'$, and hence $s = \pm s'$. □

Proposition 5.10. *The total contribution of the second Bruhat cell is*

$$\frac{\psi(\mathfrak{N})}{2^r e^{2\pi \operatorname{tr}_{\mathbb{Q}}^F (m_1+m_2)}} \prod_{j=1}^r \frac{(-4\pi i)^{\mathtt{k}_j} \sqrt{\sigma_j(m_1 m_2)}^{\mathtt{k}_j - 1}}{(\mathtt{k}_j - 2)!} \tag{33}$$

$$\times \sum_{i=1}^t \sum_{\substack{u \in U \\ \eta_i u \in F^+}} \sum_{\substack{s \in \mathfrak{b}_i \mathfrak{N}/\pm \\ s \neq 0}} \Bigg\{ \omega_{\text{fin}}(s\mathtt{b}_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u \mathtt{b}_i^{-2}; s\mathtt{b}_i^{-1}) \frac{\mathbb{N}(\eta_i u)^{1/2}}{\mathbb{N}(s)}$$

$$\times \prod_{j=1}^r J_{\mathtt{k}_j - 1}\Big(4\pi \frac{\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big) \Bigg\}.$$

5.3. **Main result.** For $\varphi \in A_{\mathtt{k}}(\mathfrak{N}, \omega)$ and $m \in \mathfrak{d}_+^{-1}$, let W_m^φ denote the Fourier coefficient of φ as in §3.4. If φ is an eigenfunction of the Hecke operator $T_{\mathfrak{n}} = R(f)$, we write

$$T_{\mathfrak{n}} \varphi = \lambda_{\mathfrak{n}}^\varphi \varphi.$$

Equating the geometric and spectral computations of the previous sections, we obtain the following upon multiplying both sides by

$$\frac{e^{2\pi \operatorname{tr}_{\mathbb{Q}}^F (m_1+m_2)}}{\psi(\mathfrak{N})} \prod_{j=1}^r \frac{(\mathtt{k}_j - 2)!}{(4\pi \sqrt{\sigma_j(m_1 m_2)})^{\mathtt{k}_j - 1}}.$$

Theorem 5.11. *Let $\mathfrak{n}$ and $\mathfrak{N}$ be integral ideals with $(\mathfrak{n}, \mathfrak{N}) = 1$. Let $\mathtt{k} = (\mathtt{k}_1, \ldots, \mathtt{k}_r)$ with all $\mathtt{k}_j > 2$. Let $\mathcal{F}$ be an orthogonal basis for $A_{\mathtt{k}}(\mathfrak{N}, \omega)$ consisting of eigenfunctions for the Hecke operator $T_{\mathfrak{n}}$. Then for any $m_1, m_2 \in \mathfrak{d}_+^{-1}$,*

$$\frac{e^{2\pi \operatorname{tr}_{\mathbb{Q}}^F (m_1+m_2)}}{\psi(\mathfrak{N})} \left[\prod_{j=1}^r \frac{(\mathtt{k}_j - 2)!}{(4\pi \sqrt{\sigma_j(m_1 m_2)})^{\mathtt{k}_j - 1}} \right] \sum_{\varphi \in \mathcal{F}} \frac{\lambda_{\mathfrak{n}}^\varphi W_{m_1}^\varphi(1) \overline{W_{m_2}^\varphi(1)}}{\|\varphi\|^2}$$

$$= \widehat{T}(m_1, m_2, \mathfrak{n}) \frac{d_F^{1/2} \mathbb{N}(\mathfrak{n})^{1/2}}{\omega_{\mathfrak{N}}(m_1/\mathtt{s})\omega_{\mathrm{fin}}(\mathtt{s})}$$

$$+ \sum_{i=1}^{t} \sum_{\substack{u \in U \\ \eta_i u \in F^+}} \sum_{\substack{s \in \mathfrak{b}_i \mathfrak{N}/\pm \\ s \neq 0}} \Bigg\{ \omega_{\mathrm{fin}}(s\mathtt{b}_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u \mathtt{b}_i^{-2}; s\mathtt{b}_i^{-1}) \frac{\mathbb{N}(\eta_i u)^{1/2}}{\mathbb{N}(s)}$$

$$\times \prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{\mathtt{k}_j}} J_{\mathtt{k}_j - 1}\Big(4\pi \frac{\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big) \Bigg\},$$

where:

- $\widehat{T}(m_1, m_2, \mathfrak{n}) \in \{0, 1\}$ *is nonzero if and only if there exists* $\mathtt{s} \in \widehat{\mathfrak{d}}^{-1}$ *such that* $m_1, m_2 \in \mathtt{s}\widehat{\mathcal{O}}$ *and* $m_1 m_2 \widehat{\mathcal{O}} = \mathtt{s}^2 \widehat{\mathfrak{n}}$ *(see also Lemma 5.2)*
- U *is a set of representatives for* $\mathcal{O}^*/\mathcal{O}^{*2}$
- $\mathtt{b}_i \widehat{\mathcal{O}} = \widehat{\mathfrak{b}}_i$ *for* $\mathtt{b}_i$ *as in* (30) *(for* $i = 1, \dots, t$*)*
- $\eta_i \in F$ *generates the principal ideal* $\mathfrak{b}_i^2 \mathfrak{n}$
- $\omega_{\mathfrak{N}} = \prod_{v|\mathfrak{N}} \omega_v \times \prod_{v \nmid \mathfrak{N}} 1.$

Remarks: (1) The above represents only part of a larger picture because we have treated just those Fourier coefficients coming from the identity component of $\overline{G}(F)\backslash\overline{G}(\mathbf{A})/K_\infty K_0(\mathfrak{N})$. The general case would result from integrating

$$K(\begin{pmatrix} \mathtt{y}_j & \\ & 1 \end{pmatrix} n_1, \begin{pmatrix} \mathtt{y}_i & \\ & 1 \end{pmatrix} n_2),$$

where $\mathtt{y}_1, \dots, \mathtt{y}_h \in \mathbf{A}_{\mathrm{fin}}^*$ are representatives for the class group of F.
(2) We can choose the basis so that each $\varphi \in V_\pi$ for some cuspidal representation π. Then the eigenvalue $\lambda_{\mathfrak{n}}^{\varphi}$ depends only on π, and not on φ. This can be seen from (38) and (39) below.

Example 5.12. *Suppose* $F = \mathbf{Q}[\sqrt{d}]$ *is a real quadratic field with narrow class number* 1 *(e.g.* $d = 2, 5$*).*

For such fields, the fundamental unit ε has negative norm. We can take $U = \{\pm 1, \pm\varepsilon\}$, and $\mathfrak{n} = (\eta)$, with $\eta \in F^+$. See Theorem 7.2 below.

Example 5.13. *Let* $F = \mathbf{Q}[\sqrt{d}]$ *be a real quadratic field with class number* 1 *and narrow class number* 2 *(e.g.* $d = 3$*).*

In this case, the fundamental unit ε has positive norm. If $\mathfrak{n}$ does not have a totally positive generator (i.e. if $\mathfrak{n} = (\eta)$ is nontrivial in the narrow class group), then there is no $u \in U$ for which $\eta u \in F^+$. Thus for such $\mathfrak{n}$, the entire Kloosterman term in the above theorem vanishes.

For general F, η_i can only make a nontrivial contribution to the Kloosterman term if it satisfies the sign condition

$$(\operatorname{sgn} \eta_i^{\sigma_1}, \dots, \operatorname{sgn} \eta_i^{\sigma_r}) = \pm(\operatorname{sgn} \varepsilon^{\sigma_1}, \dots, \operatorname{sgn} \varepsilon^{\sigma_r})$$

for some unit ε in a fundamental unit system. We now give some examples to illustrate various possibilities for the ideals $\mathfrak{b}_i$.

Example 5.14. *F has class number* 2.

In the class group $\mathrm{Cl}(F)$, the equation $[\mathfrak{b}]^2[\mathfrak{n}] = 1$ has solutions if and only if $[\mathfrak{n}]$ is trivial (since $x^2 = 1$ for all $x \in \mathrm{Cl}(F)$). Hence the Kloosterman term is nonzero only if $\mathfrak{n} = (\eta)$ is a principal ideal. In this case, we can take $\mathfrak{b}_1$ to be any non-principal ideal and $\mathfrak{b}_2 = \mathcal{O}$.

Example 5.15. *F has class number 3.*

Now we have $x^2 = x^{-1}$ for all $x \in \mathrm{Cl}(F)$. Therefore the equation $[\mathfrak{b}]^2[\mathfrak{n}] = 1$ has a unique solution in $\mathrm{Cl}(F)$, and we can take $\mathfrak{b} = \mathfrak{n}$.

Example 5.16. *F has odd class number.*

Suppose $h = |\,\mathrm{Cl}(F)|$ is odd. Then the equation $[\mathfrak{b}]^2[\mathfrak{n}] = 1$ has a unique solution. For example we can take $\mathfrak{b} = \mathfrak{n}^{(h-1)/2}$.

6. Weighted distribution of Hecke eigenvalues

As an application of Theorem 5.11, we will show that relative to a certain measure, the (normalized) eigenvalues of the Hecke operator $T_{\mathfrak{p}}$ have a weighted equidistribution in the interval $[-2, 2]$ as $\mathbb{N}(\mathfrak{N}) \to \infty$.

6.1. **Estimates.** As a function of $\mathfrak{N}$, the contribution of the first Bruhat cell (given in Proposition 5.4) has order

$$\psi(\mathfrak{N}) = [K_{\mathrm{fin}} : K_0(\mathfrak{N})] = \mathbb{N}(\mathfrak{N}) \prod_{\mathfrak{p}|\mathfrak{N}} (1 + \frac{1}{\mathbb{N}\mathfrak{p}}).$$

Here we will show that this is the dominant term in the Petersson trace formula as $\mathbb{N}(\mathfrak{N}) \to \infty$. For this, we need to show that the contribution (33) of the second Bruhat cell is small in comparison.

We start with the following naive estimate (essentially the triangle inequality) for the Kloosterman sums.

Lemma 6.1. *For any $m_1, m_2 \in \mathfrak{d}^{-1}$, nonzero $\mathtt{n} \in \widehat{\mathcal{O}}$ and $\mathtt{c} \in \widehat{\mathfrak{N}} \cap \mathbf{A}^*_{\mathrm{fin}}$,*

$$|S_{\omega_{\mathfrak{N}}}(m_1, m_2; \mathtt{n}; \mathtt{c})| \le \mathbb{N}(\mathtt{n})\mathbb{N}(\mathtt{c}).$$

Proof. It suffices to prove the lemma locally. Note that

$$\mathbb{N}(\mathtt{c}) = \prod_{v<\infty} \mathbb{N}(\mathtt{c}_v) = \prod_{v<\infty} |\mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v|.$$

We have

$$|S_{\omega_{\mathfrak{N},v}}(m_1, m_2; \mathtt{n}_v; \mathtt{c}_v)| \le \sum_{\substack{s_1,s_2 \in \mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v \\ s_1 s_2 \equiv \mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v}} |\theta_v(\frac{m_1 s_1 + m_2 s_2}{\mathtt{c}_v})\omega_{\mathfrak{N},v}(s_2)^{-1}| = \sum_{\substack{s_1,s_2 \in \mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v \\ s_1 s_2 \equiv \mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v}} 1.$$

In the case where $\mathrm{ord}_v(\mathtt{n}_v) > \mathrm{ord}_v(\mathtt{c}_v)$, we can replace $\mathtt{n}_v$ by $\mathtt{c}_v$ without loss of generality. Thus assuming $0 \le \mathrm{ord}_v(\mathtt{n}_v) \le \mathrm{ord}_v(\mathtt{c}_v)$, we need to count the elements of the following set:

$$\{(s_1, s_2) \in (\mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v)^2 |\, s_1 s_2 \equiv \mathtt{n}_v \bmod \mathtt{c}_v\mathcal{O}_v\}. \tag{34}$$

For a fixed $s_1 \in \mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v$ (we can assume that $0 \le \mathrm{ord}_v(s_1) \le \mathrm{ord}_v(\mathtt{c}_v)$), there exists a solution s_2 to the congruence if and only if $\mathrm{ord}_v(s_1) \le \mathrm{ord}_v(\mathtt{n}_v)$. If this condition holds, then the number of solutions s_2 is

$$|\frac{\mathtt{c}_v}{s_1}\mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v| = |\mathcal{O}_v/s_1\mathcal{O}_v| = \mathbb{N}(s_1).$$

Thus the cardinality of (34) is

$$\sum_{\substack{s\in\mathcal{O}_v/\mathtt{c}_v\mathcal{O}_v\\ \mathrm{ord}_v(s)\le\mathrm{ord}_v(\mathtt{n}_v)}} \mathbb{N}(s) = \sum_{\ell=0}^{\mathrm{ord}_v \mathtt{n}_v} \mathbb{N}(\varpi_v^\ell)\cdot|(\mathcal{O}_v/\frac{\mathtt{c}_v}{\varpi_v^\ell}\mathcal{O}_v)^*|$$

$$\le \sum_{\ell=0}^{\mathrm{ord}_v \mathtt{n}_v} \mathbb{N}(\varpi_v^\ell)\mathbb{N}(\mathtt{c}_v/\varpi_v^\ell) = \mathbb{N}(\mathtt{c}_v)(\mathrm{ord}_v(\mathtt{n}_v)+1) \le \mathbb{N}(\mathtt{c}_v)\mathbb{N}(\mathtt{n}_v) \tag{35}$$

as desired. □

Proposition 6.2. *As a function of* $\mathfrak{N}$, *the contribution* (33) *of the second Bruhat cell is* $\ll \frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}}$ *for any* $0<\varepsilon<1$ *as* $\mathbb{N}(\mathfrak{N})\to\infty$. *Here the implied constant depends on* $\mathfrak{n}$, $\mathtt{k}$ *and* ε.

Remark: The middle expression of (35) is $\ll_{\varepsilon'} \mathbb{N}(\mathtt{c}_v)\mathbb{N}(\mathtt{n}_v)^{\varepsilon'}$ for any $\varepsilon'>0$. Using this, the dependence of (33) on $\mathfrak{n}$ can easily be shown to be $\ll \mathbb{N}(\mathfrak{n})^{3/2+\varepsilon'-\varepsilon/2}$ for $0<\varepsilon<1$.

Proof. By Lemma 6.1, (33) is

$$\ll \psi(\mathfrak{N})\sum_{i=1}^{t}\sum_{\substack{u\in U\\ \eta_i u\in F^+}}\sum_{\substack{s\in\mathfrak{b}_i\mathfrak{N}/\pm\\ s\neq0}} \frac{\mathbb{N}(\eta_i u)\mathbb{N}(s)}{\mathbb{N}(\mathfrak{b}_i)^3}\frac{\mathbb{N}(\eta_i u)^{1/2}}{\mathbb{N}(s)}\prod_{j=1}^{r} J_{\mathtt{k}_j-1}(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|})$$

$$\ll \psi(\mathfrak{N})\sum_{i=1}^{t}\sum_{\substack{u\in U\\ \eta_i u\in F^+}}\sum_{\substack{s\in\mathfrak{b}_i\mathfrak{N}/\mathcal{O}^*\\ s\neq0}}\sum_{a\in\mathcal{O}^*/\{\pm1\}}\prod_{j=1}^{r} J_{\mathtt{k}_j-1}(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)||\sigma_j(a)|}).$$

We remark that $|\sigma_j(s)|$ is not well-defined for $s\in\mathfrak{b}_i\mathfrak{N}/\mathcal{O}^*$, so the summands depend on a choice of representatives s, which we regard as fixed. By the units theorem, the set

$$\{(\log|\sigma_1(a)|,\ldots,\log|\sigma_r(a)|) : a\in\mathcal{O}^*\}$$

is a full lattice Λ in the $(r-1)$-dimensional hyperplane $x_1+\cdots+x_r=0$ in $\mathbf{R}^r$. Thus the summation over $a\in\mathcal{O}^*/\{\pm1\}$ can be replaced by a sum over $\lambda=(\lambda_1,\ldots,\lambda_r)\in\Lambda$. Recall that

$$J_{\mathtt{k}_j-1}(x) \ll \min(|x|^{-1/2},|x|^{\mathtt{k}_j-1}) \le \frac{1}{\max(|x|^{1/2},|x|^{-2})} \le \frac{2}{|x|^{1/2}+|x|^{-2}}.$$

As a result, if we let $\tau_j = \log\left|\frac{\sigma_j(s)}{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}\right|$ and $\lambda_j=\log|\sigma_j(a)|$, we can bound $J_{\mathtt{k}_j-1}(e^{-(\lambda_j+\tau_j)})$ to give

$$\sum_{a\in\mathcal{O}^*/\{\pm1\}}\prod_{j=1}^{r} J_{\mathtt{k}_j-1}(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)||\sigma_j(a)|}) \ll \sum_{\lambda\in\Lambda}\frac{1}{\prod_{j=1}^{r}(e^{-(\lambda_j+\tau_j)/2}+e^{2(\lambda_j+\tau_j)})}.$$

Let $\beta = 2 - \varepsilon$ for $0 < \varepsilon < 1$. Using $\sum_j \lambda_j = 0$, the right-hand sum is

$$= \sum_{\lambda \in \Lambda} \frac{1}{\prod_{j=1}^{r} e^{\beta(\lambda_j + \tau_j)} \prod_{j=1}^{r} (e^{-(1/2+\beta)(\lambda_j+\tau_j)} + e^{\varepsilon(\lambda_j+\tau_j)})}$$

$$= \sum_{\lambda \in \Lambda} \frac{1}{e^{(\sum_{j=1}^{r} \tau_j)\beta} \prod_{j=1}^{r} (e^{-(1/2+\beta)(\lambda_j+\tau_j)} + e^{\varepsilon(\lambda_j+\tau_j)})}$$

$$= \left| \frac{(4\pi)^r \sqrt{\mathbb{N}(\eta_i u m_1 m_2)}}{\mathbb{N}(s)} \right|^{\beta} \sum_{\lambda \in \Lambda} \frac{1}{\prod_{j=1}^{r} (e^{-(1/2+\beta)(\lambda_j+\tau_j)} + e^{\varepsilon(\lambda_j+\tau_j)})}.$$

We claim that this last sum over λ is bounded, independently of s. Let L_0 be the hyperplane $x_1 + \cdots + x_r = 0$ containing Λ. Then

$$\sum_{\lambda \in \Lambda} \frac{1}{\prod_{j=1}^{r} (e^{-(1/2+\beta)(\lambda_j+\tau_j)} + e^{\varepsilon(\lambda_j+\tau_j)})} \ll \int_{L_0} \frac{dx_1 \cdots dx_{r-1}}{\prod_{j=1}^{r} (e^{-(1/2+\beta)(x_j+\tau_j)} + e^{\varepsilon(x_j+\tau_j)})}.$$

The r^{th} factor of the integrand is bounded, so the above is

$$\ll \int_{\mathbf{R}^{r-1}} \frac{dx_1 \cdots dx_{r-1}}{\prod_{j=1}^{r-1} (e^{-(1/2+\beta)(x_j+\tau_j)} + e^{\varepsilon(x_j+\tau_j)})} = \prod_{j=1}^{r-1} \int_{\mathbf{R}} \frac{dx}{e^{-(1/2+\beta)x} + e^{\varepsilon x}} < \infty$$

as claimed. Therefore (33) is

$$\ll \psi(\mathfrak{N}) \sum_{i=1}^{t} \sum_{\substack{u \in U \\ \eta_i u \in F^+}} \sum_{\substack{s \in \mathfrak{b}_i \mathfrak{N}/\mathcal{O}^* \\ s \neq 0}} \left| \frac{(4\pi)^r \sqrt{\mathbb{N}(\eta_i u m_1 m_2)}}{\mathbb{N}(s)} \right|^{2-\varepsilon}.$$

The first two sums are taken over finite sets. The set of $s \in \mathfrak{b}_i \mathfrak{N}/\mathcal{O}^*$ is in 1-1 correspondence with the set of principal ideals (s) divisible by $\mathfrak{b}_i \mathfrak{N}$. Hence we can replace the sum over s by a larger set: all integral ideals $\mathfrak{a}$ divisible by $\mathfrak{N}$. Therefore (33) is

$$\ll \psi(\mathfrak{N}) \sum_{\mathfrak{N}|\mathfrak{a}} \frac{1}{\mathbb{N}(\mathfrak{a})^{2-\varepsilon}} \ll \frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}} \sum_{\mathfrak{a}'} \frac{1}{\mathbb{N}(\mathfrak{a}')^{2-\varepsilon}} \ll \frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}}.$$

(We use the absolute convergence of the zeta function $\zeta_F(s) = \sum_{\mathfrak{a}'} \frac{1}{\mathbb{N}(\mathfrak{a}')^s}$ for $\operatorname{Re} s > 1$.) □

6.2. **Weighted equidistribution.** Let (X, μ) be a Borel measure space. For each $i = 1, 2, \ldots$ let $\mathcal{F}_i$ be a finite nonempty index set, and let $S_i = \{x_{ij}\}_{j \in \mathcal{F}_i}$ be a finite sequence of points of X. Suppose each $j \in \mathcal{F}_i$ is assigned a weight $w_{ij} \in \mathbf{R}^+$. Define

$$d\mu_i = \frac{\sum_{j \in \mathcal{F}_i} w_{ij} \delta_{x_{ij}}}{\sum_{j \in \mathcal{F}_i} w_{ij}},$$

where $\delta_{x_{ij}}$ is the Dirac measure at x_{ij}. We say that the sequence $\{S_i\}$ is w-**equidistributed** with respect to the measure $d\mu$ if

$$\lim_{i \to \infty} d\mu_i = \lim_{i \to \infty} \frac{\sum_{j \in \mathcal{F}_i} w_{ij} \delta_{x_{ij}}}{\sum_{j \in \mathcal{F}_i} w_{ij}} = d\mu.$$

This means that for any continuous function $f : X \to \mathbf{C}$, we have

$$\lim_{i \to \infty} \int_X f(x) d\mu_i(x) = \lim_{i \to \infty} \frac{\sum_{j \in \mathcal{F}_i} w_{ij} f(x_{ij})}{\sum_{j \in \mathcal{F}_i} w_{ij}} = \int_X f(x) d\mu(x).$$

If $w_{ij} = 1$ for all i, j, then this definition reduces to that of equidistribution given in §1 of [Se].

6.3. **The distribution theorem.** Let $\mathfrak{p}$ be a prime ideal, not dividing the level $\mathfrak{N}$. In this section we will apply the main theorem in the case where $\mathfrak{n} = \mathfrak{p}^\ell$ ($\ell \geq 0$), and $m = m_1 = m_2 \in \mathfrak{d}_+^{-1}$.

For each irreducible summand π of $H_{\mathbf{k}}(\mathfrak{N}, \omega)$ (cf. (7)), let $\mathcal{F}_\pi$ be an orthogonal basis for the nonzero finite-dimensional subspace $\pi \cap A_{\mathbf{k}}(\mathfrak{N}, \omega)$. Let

$$\mathcal{F} = \bigcup_\pi \mathcal{F}_\pi \tag{36}$$

be the resulting orthogonal basis for $A_{\mathbf{k}}(\mathfrak{N}, \omega)$.

Lemma 6.3. *For any $\varphi \in \mathcal{F}$ and any prime ideal $\mathfrak{p} \nmid \mathfrak{N}$, the cuspidal representation (π, V) containing φ is unramified at $\mathfrak{p}$. Furthermore, φ is an eigenfunction of the global Hecke operator $T_{\mathfrak{p}^\ell}$, and the associated eigenvalue $\lambda^\varphi_{\mathfrak{p}^\ell}$ coincides with the local eigenvalue $\lambda_{\mathfrak{p}_v^\ell}$ attached to $\pi_\mathfrak{p}$ in Prop. 4.4.*

Proof. Write $\pi_{\text{fin}} = \pi_\mathfrak{p} \otimes \pi'$, where $\pi' = \otimes_{\substack{v<\infty \\ v \neq v_\mathfrak{p}}} \pi_v$ is a representation of the restricted direct product $G' = \prod'_{\substack{v<\infty \\ v\neq v_\mathfrak{p}}} G_v$. Let $K' = G' \cap K_1(\mathfrak{N})$. Let $f = f_\mathfrak{n}$ with $\mathfrak{n} = (1)$. Then $\pi_{\text{fin}}(f)$ is the projection operator of V_{fin} onto $\pi_{\text{fin}}^{K_1(\mathfrak{N})}$, and

$$\pi_{\text{fin}}^{K_1(\mathfrak{N})} = \pi_{\text{fin}}(f) V_{\text{fin}} = \pi_\mathfrak{p}(f_{v_\mathfrak{p}}) V_\mathfrak{p} \otimes \pi'(f') V' = \pi_\mathfrak{p}^{K_\mathfrak{p}} \otimes \pi'^{K'}. \tag{37}$$

The middle equality holds e.g. by Prop. 13.17 of [KL2].

Now take $\mathfrak{n} = \mathfrak{p}^\ell$ and $f = f_\infty \times f_\mathfrak{n}$. By the definition of $A_{\mathbf{k}}(\mathfrak{N}, \omega)$ (eq. (8)) we can write $\varphi = w_\infty \otimes w_{\text{fin}}$, where $w_\infty = \otimes v_{\pi_{\infty_j}}$ and $w_{\text{fin}} \in \pi_{\text{fin}}^{K_1(\mathfrak{N})}$. Because $0 \neq w_{\text{fin}} \in \pi_{\text{fin}}^{K_1(\mathfrak{N})}$, it follows immediately from (37) that $\pi_\mathfrak{p}$ is unramified. Because $\pi_\mathfrak{p}$ is also unitary, it follows that $\pi_\mathfrak{p} = \pi_\chi$ is induced from some unramified character $\chi(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)) = \chi_1(a)\chi_2(d)$ of $B(F_v)$. Therefore $\pi_\mathfrak{p}^{K_\mathfrak{p}} = \mathbf{C}\phi_0$ is one-dimensional (for notation see (18)), and we can write $w_{\text{fin}} = \phi_0 \otimes w'$ as in (37). Thus $\varphi = w_\infty \otimes \phi_0 \otimes w'$, and

$$\begin{aligned} T_{\mathfrak{p}^\ell}\varphi = R(f)\varphi &= \pi_\infty(f_\infty) w_\infty \otimes \pi_\mathfrak{p}(f_{v_\mathfrak{p}})\phi_0 \otimes \pi'(f')w' \\ &= w_\infty \otimes \lambda_{\mathfrak{p}_v^\ell}\phi_0 \otimes w' = \lambda_{\mathfrak{p}_v^\ell}\varphi. \end{aligned}$$

□

The local Langlands class of $\pi_\mathfrak{p}$ is the $\mathrm{GL}_2(\mathbf{C})$-conjugacy class of the matrix $g(\pi_\mathfrak{p}) = \left(\begin{smallmatrix} \chi_1(\varpi_\mathfrak{p}) & \\ & \chi_2(\varpi_\mathfrak{p}) \end{smallmatrix}\right)$. Let $q = \mathbb{N}(\mathfrak{p})$. By the above lemma and Prop. 4.4, the trace of $g(\pi_\mathfrak{p})$ is given by

$$\chi_1(\varpi_\mathfrak{p}) + \chi_2(\varpi_\mathfrak{p}) = q^{-1/2}\lambda_\mathfrak{p}^\varphi. \tag{38}$$

If χ is unitary (i.e. $\pi_\mathfrak{p}$ is not complementary series), then it is clear that $|\lambda_\mathfrak{p}^\varphi| \leq 2q^{1/2}$.

In fact by the Ramanujan conjecture this is always the case.[1]

When $\mathfrak{n} = \mathfrak{p}^\ell$, the operator $\omega_\mathfrak{p}(\varpi_\mathfrak{p})^{-\ell/2}R(f)$ is self-adjoint (cf. Prop. 4.3), so its eigenvalues are real numbers. We let

$$\nu^\varphi_{\mathfrak{p}^\ell} \stackrel{\text{def}}{=} \omega_\mathfrak{p}(\varpi_\mathfrak{p})^{-\ell/2} q^{-\ell/2} \lambda^\varphi_{\mathfrak{p}^\ell} \in \mathbf{R}$$

denote this normalized Hecke eigenvalue. We sometimes write $\nu^\pi_{\mathfrak{p}^\ell}$ to emphasize that it depends only on the cuspidal representation π. Note that $\nu^\varphi_\mathfrak{p} \in [-2, 2]$ by the Ramanujan conjecture. By Prop. 4.5,

$$\nu^\varphi_{\mathfrak{p}^\ell} = X_\ell(\nu^\varphi_\mathfrak{p}). \tag{39}$$

We will adapt the argument of [Li1] for finding the asymptotic weighted distribution of the set of $\nu^\varphi_\mathfrak{p}$.

Lemma 6.4. *In the notation of Theorem 5.11, for any $\ell \geq 0$ and any $m \in \mathfrak{d}^{-1}_+$, $\widehat{T}(m, m, \mathfrak{p}^\ell) \neq 0$ if and only if both of the following hold:*

(1) *$\ell = 2\ell'$ is even*

(2) *$0 \leq \ell' \leq \operatorname{ord}_\mathfrak{p}(m\mathfrak{d})$.*

Proof. The two conditions of Lemma 5.2 specialize respectively to the two conditions above in the special case where $m_1 = m_2 = m$ and $\mathfrak{n} = \mathfrak{p}^\ell$. □

Proposition 6.5. *Fix $m \in \mathfrak{d}^{-1}_+$ and a prime ideal $\mathfrak{p} \nmid \mathfrak{N}$. For each $\varphi \in \mathcal{F}$, define a weight $w_\varphi = \frac{|W^\varphi_m(1)|^2}{\|\varphi\|^2}$. Then for any $\ell \geq 0$ and $0 < \varepsilon < 1$,*

$$\sum_{\varphi \in \mathcal{F}} X_\ell(\nu^\varphi_\mathfrak{p}) w_\varphi = \begin{cases} J\psi(\mathfrak{N}) + O(\frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}}) & \text{if } \ell = 2\ell' \text{ with } 0 \leq \ell' \leq \operatorname{ord}_v m\mathfrak{d} \\ O(\frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}}) & \text{otherwise,} \end{cases}$$

where X_ℓ is the Chebyshev polynomial defined in Prop. 4.5, and

$$J = \frac{d_F^{1/2}}{e^{4\pi \operatorname{tr}^F_\mathbf{Q}(m)}} \prod_{j=1}^r \frac{(4\pi\sigma_j(m))^{\mathtt{k}_j - 1}}{(\mathtt{k}_j - 2)!}.$$

The implied constant depends only on m, $\mathfrak{p}$, ℓ, $\mathtt{k}$ and ε.

Remark: This shows in particular that when $\mathbb{N}(\mathfrak{N})$ is sufficiently large,

(1) $A_\mathtt{k}(\mathfrak{N}, \omega)$ is nontrivial

(2) $W^\varphi_m(1)$ is nonzero for some $\varphi \in \mathcal{F}$.

Proof. This follows from the generalized Petersson trace formula. We use the form developed in the proof, rather than the final statement in Theorem 5.11. The spectral side (23) of the trace formula with $\mathfrak{n} = \mathfrak{p}^\ell$ and $m_1 = m_2 = m \in \mathfrak{d}^{-1}_+$ gives

$$\sum_{\varphi \in \mathcal{F}} \frac{\lambda^\varphi_{\mathfrak{p}^\ell} |W^\varphi_m(1)|^2}{\|\varphi\|^2} = \sum_{\varphi \in \mathcal{F}} \omega_\mathfrak{p}(\varpi_\mathfrak{p})^{\ell/2} q^{\ell/2} X_\ell(\nu^\varphi_\mathfrak{p}) w_\varphi.$$

[1] The Ramanujan conjecture was proven at all but a finite number of unspecified places for holomorphic cuspidal representations π of $\mathrm{GL}_2(\mathbf{A})$ with all weights ≥ 2 by Brylinski and Labesse ([BL], Theorem 3.4.6). Recently the full conjecture (at all places, when all weights are ≥ 2) was proven by Blasius, with a parity condition on the weights [Bl]. The parity requirement was removed in the thesis of his student L. Nguyen [Ng]. However, as we remark after Theorem 6.6 below, our results do not actually depend on this deep theorem.

The above is equal to the geometric side, which by Prop. 5.4 and Prop. 6.2 is

$$= \frac{\widehat{T}(m,m,\mathfrak{p}^{\ell})}{\omega_{\mathfrak{N}}(m/\mathtt{s})\omega_{\mathrm{fin}}(\mathtt{s})} q^{\ell/2} J\psi(\mathfrak{N}) + O(\frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}})$$

for J as above. By the above lemma, the first term is nonzero if and only if $\ell = 2\ell'$ with $0 \le \ell' \le \mathrm{ord}_{\mathfrak{p}}(m\mathfrak{d})$. In this case we can take

$$\mathtt{s} = (\ldots, m, m, \overset{\mathfrak{p}^{\mathrm{th}}}{m\varpi_{\mathfrak{p}}^{-\ell'}}, m, m, \ldots).$$

Note that $\omega_{\mathfrak{N}}(m/\mathtt{s}) = \prod_{\mathfrak{p}|\mathfrak{N}} \omega_{\mathfrak{p}}(1) = 1$, so

$$\omega_{\mathfrak{N}}(m/\mathtt{s})\omega_{\mathrm{fin}}(\mathtt{s}) = \omega_{\mathrm{fin}}(\mathtt{s}) = \omega_{\mathrm{fin}}(m)\omega_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^{-\ell'}.$$

Furthermore $\omega_{\mathrm{fin}}(m) = \omega_{\infty}(m)^{-1} = 1$ since $m \in F^{+}$. Thus in this case the geometric side is

$$= \omega_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^{\ell/2} q^{\ell/2} J\psi(\mathfrak{N}) + O(\frac{\psi(\mathfrak{N})}{\mathbb{N}(\mathfrak{N})^{2-\varepsilon}}).$$

The proposition now follows by equating the spectral side with the geometric side and dividing by $\omega_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^{\ell/2} q^{\ell/2}$. □

Theorem 6.6. *Let* $\mathfrak{p}$ *be a prime ideal of* F*, and let* $\mathtt{k} = (\mathtt{k}_1, \ldots, \mathtt{k}_r)$ *be a weight vector with all* $\mathtt{k}_j > 2$*. For each* $i = 1, 2, \ldots$

- *let* $\mathfrak{N}_i$ *be an ideal coprime to* $\mathfrak{p}$*, with* $\lim_{i\to\infty} \mathbb{N}(\mathfrak{N}_i) = \infty$*,*
- *let* ω_i *be a unitary character as in §3.3 relative to* $\mathfrak{N}_i$ *and* $\mathtt{k}$*,*
- *let* $\mathcal{F}_i$ *be an orthogonal basis for* $A_{\mathtt{k}}(\mathfrak{N}_i, \omega_i)$ *as in* (36)*.*

Fix any $m \in \mathfrak{d}_{+}^{-1}$*, and define weights* $w_{\varphi} = |W_m^{\varphi}(1)|^2/\|\varphi\|^2$ *for* $\varphi \in \mathcal{F}_i$*. For each* i*, define a sequence*

$$S_i = \{\nu_{\mathfrak{p}}^{\varphi}\}_{\varphi \in \mathcal{F}_i}$$

in the interval $[-2, 2]$*. Then the sequence* S_i *is* w_{φ}*-equidistributed relative to the measure*

$$d\mu(x) = \sum_{\ell'=0}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}m)} X_{2\ell'}(x) d\mu_{\infty}(x),$$

where $d\mu_{\infty}(x)$ *is the Sato-Tate measure defined in the Introduction. In other words, for any continuous function* h *on* $\mathbf{R}$*,*

$$\lim_{i\to\infty} \frac{\sum_{\varphi\in\mathcal{F}_i} h(\nu_{\mathfrak{p}}^{\varphi}) w_{\varphi}}{\sum_{\varphi\in\mathcal{F}_i} w_{\varphi}} = \int_{\mathbf{R}} h(x) d\mu(x).$$

Remarks: (1) When $\mathfrak{p} \nmid m\mathfrak{d}$, the measure μ coincides with μ_{∞} and is independent of $\mathfrak{p}$. Further taking all $\mathtt{k}_j$ even and ω_i trivial, we immediately obtain Theorem 1.1.

(2) The above result (and its proof) is actually independent of the Ramanujan conjecture. All we need is the existence of a finite interval $I_{\mathfrak{p}}$ which contains all of the eigenvalues $\nu_{\mathfrak{p}}^{\varphi}$. This is elementary ([Ro], Prop. 2.9).

(3) The theorem shows in particular that the Satake traces $\nu_{\mathfrak{p}}^{\pi}$ are dense in the interval $[-2, 2]$. The referee has pointed out that this can also be seen directly by considering CM cusp forms.

Proof. Setting $\ell = 0$ in the previous proposition, we have

$$\sum_{\varphi \in \mathcal{F}_i} w_\varphi = J\psi(\mathfrak{N}_i) + O(\frac{\psi(\mathfrak{N}_i)}{\mathbb{N}(\mathfrak{N}_i)^{2-\varepsilon}}).$$

Therefore for any $\ell \geq 0$

$$\lim_{i\to\infty} \frac{\sum_{\varphi\in\mathcal{F}_i} X_\ell(\nu_\mathfrak{p}^\varphi) w_\varphi}{\sum_{\varphi\in\mathcal{F}_i} w_\varphi} = \begin{cases} 1 & \text{when } \ell = 2\ell', \text{ with } 0 \leq \ell' \leq \operatorname{ord}_v m\mathfrak{d}, \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{\mathbf{R}} X_\ell(x) d\mu(x). \tag{40}$$

This last equality holds by the orthonormality of the polynomials $X_n(x)$

$$\int_{\mathbf{R}} X_i(x) X_j(x) d\mu_\infty(x) = \delta_{ij}$$

(cf. [Se]). Because $\deg X_\ell = \ell$, the set $\{X_\ell\}$ spans the space of all polynomials. Because the space of polynomials is dense in $L^\infty([-2,2])$, we can replace X_ℓ by any continuous function h in (40) and thus obtain the result (see §29.3 of [KL2] for details). □

7. Variants and special cases

We have four corresponding types of parameters (under various hypotheses):

$$\begin{matrix} \text{Satake} & \longleftrightarrow & \text{Hecke} & \longleftrightarrow & \text{Whittaker} & \longleftrightarrow & \text{Classical} \\ \nu_\mathfrak{p}^\pi & & \lambda_\mathfrak{n}^\varphi & & W_m^\varphi(y) & & a_m(h). \end{matrix}$$

For convenience, we give the correspondences explicitly here, so that when possible anyone can rewrite the spectral terms $\frac{\lambda_\mathfrak{n}^\varphi W_{m_1}^\varphi(1)\overline{W_{m_2}^\varphi(1)}}{\|\varphi\|^2}$ in the main formula purely in terms of their parameter of choice. Let $\mathtt{n}\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}$ and $\mathtt{d}\widehat{\mathcal{O}} = \widehat{\mathfrak{d}}$. If $W_m^\varphi(1/\mathtt{d}) = 1$, then by Corollaries 4.7 and 4.8,

$$\lambda_\mathfrak{n}^\varphi = \mathbb{N}(\mathfrak{n}) W_1^\varphi(\mathtt{n}/\mathtt{d}) \qquad \text{and} \qquad W_m^\varphi(1) = \frac{e^{2\pi r}\prod_{j=1}^r \sigma_j(m)^{\mathtt{k}_j/2-1}}{d_F e^{2\pi \operatorname{tr}(m)}} \lambda_{m\mathfrak{d}},$$

either of which can be used if an orthogonal basis of such φ is given. Also, using (39) we have

$$\lambda_\mathfrak{n}^\varphi = \omega_{\text{fin}}(\mathtt{n})^{1/2} \mathbb{N}(\mathfrak{n})^{1/2} \prod_{\mathfrak{p}|\mathfrak{n}} X_{\operatorname{ord}_\mathfrak{p}(\mathfrak{n})}(\nu_\mathfrak{p}^\pi). \tag{41}$$

The classical picture is given explicitly at the end of this section.

The case of narrow class number 1. From now on we assume that F has narrow class number 1, i.e. that every fractional ideal in F has a totally positive generator. This implies that every totally positive unit is the square of a unit ([CH], Lemma 11.6). In this case Theorem 5.11 simplifies substantially, and a classical interpretation follows from (14).

Fix $\eta, N, d \in F^+$ such that

$$(\eta) = \mathfrak{n}, \quad (N) = \mathfrak{N}, \quad (d) = \mathfrak{d}. \tag{42}$$

Proposition 7.1. *Suppose F has narrow class number 1. For $\delta = \begin{pmatrix} m_2/m_1 & \\ & 1\end{pmatrix}$,*

$$I_\delta(f) = \frac{T(dm_1, dm_2, \eta) d_F^{1/2} \mathbb{N}(\eta)^{1/2} \psi(\mathfrak{N})}{e^{2\pi \operatorname{tr}_{\mathbf{Q}}^F(m_1+m_2)} \omega_{\mathfrak{N}}(\sqrt{m_1\eta/m_2})} \left[\prod_{j=1}^r \frac{\left(4\pi\sqrt{\sigma_j(m_1m_2)}\right)^{\mathtt{k}_j - 1}}{(\mathtt{k}_j-2)!\operatorname{sgn}(\sigma_j(\sqrt{\frac{m_1m_2}{\eta}}))^{\mathtt{k}_j}} \right],$$

where

$$T(a_1,a_2,a_3) = \begin{cases} 1 & \text{if } a_ia_j/a_k \text{ is a square in } \mathcal{O} \text{ for all} \\ & \text{distinct } i,j,k \in \{1,2,3\} \\ 0 & \text{otherwise,} \end{cases}$$

and the square roots are chosen compatibly so that $\sqrt{m_1m_2/\eta}\sqrt{m_1\eta/m_2} = m_1$.

Proof. By Prop. 5.4, we know that I_δ is nonzero only if

- $\frac{m_1}{\mathtt{s}}, \frac{m_2}{\mathtt{s}} \in \widehat{\mathcal{O}}$ for some $\mathtt{s} \in \widehat{\mathfrak{d}}^{-1}$
- $\frac{m_1m_2}{\mathtt{s}^2}\widehat{\mathcal{O}} = \widehat{\mathfrak{n}}$.

If these hold, then because the class number is 1, we can actually take $\mathtt{s} \in \mathfrak{d}^{-1}$, so we change fonts to s as a reminder that $s \in F$. Furthermore the second condition is equivalent to $m_1m_2 = s^2u\eta$ for some unit u. Because $m_1m_2, s^2, \eta \in F^+$, it follows that $u \in F^+$, and hence u is the square of a unit. We can absorb this unit into s, so the condition becomes

$$m_1m_2 = s^2\eta. \tag{43}$$

Write $s = s'/d$ for d as in (42). Then the above conditions are equivalent to

(1) $\frac{dm_1}{s'}, \frac{dm_2}{s'} \in \mathcal{O}$ for some $s' \in \mathcal{O}$
(2) $\frac{(dm_1)(dm_2)}{s'^2} = \eta$.

It is easy to check that these conditions hold if and only if $T(dm_1, dm_2, \eta) = 1$.

Now suppose the above holds, so that by Prop. 5.4,

$$I_\delta = \left[\prod_{j=1}^r \frac{\left(4\pi\sqrt{\sigma_j(m_1m_2)}\right)^{\mathtt{k}_j-1}}{(\mathtt{k}_j-2)!} \right] \frac{d_F^{1/2}\mathbb{N}(\mathfrak{n})^{1/2}\psi(\mathfrak{N})}{e^{2\pi \operatorname{tr}_{\mathbf{Q}}^F(m_1+m_2)}\omega_{\mathfrak{N}}(m_1/s)\omega_{\text{fin}}(s)}.$$

By (43), we can write $s = \sqrt{m_1m_2/\eta}$. This element is defined up to ± 1, and there is no canonical choice since we cannot guarantee that $s \in F^+$. (E.g. $(1+\sqrt{2})^2$ has no totally positive square root.) In any case, the final result is independent of this choice by the remark on page 16. Using this expression for s, the proposition follows by the fact that

$$\omega_{\text{fin}}(s) = \omega_\infty(s)^{-1} = \prod_{j=1}^r \operatorname{sgn}(\sigma_j(s))^{\mathtt{k}_j}.$$

□

For the second type of I_δ (with $\delta = \begin{pmatrix} & \mu \\ 1 & \end{pmatrix}$), the condition on μ becomes $\mu = -\eta u/c^2$ for some totally positive unit $u \in U$ and nonzero $c \in N\mathcal{O}$. As in the previous case, u is a square, so it can be absorbed into c. Thus

$$\mu = -\eta/c^2$$

for some $c \in \mathfrak{N}$. Under this condition, by Prop. 5.7 the contribution is

$$\begin{aligned} I_\delta = & \frac{\psi(\mathfrak{N})}{2^r e^{2\pi \operatorname{tr}_{\mathbf{Q}}^F(m_1+m_2)}} \omega_{\text{fin}}(c) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta; c) \frac{\mathbb{N}(\eta)^{1/2}}{\mathbb{N}(c)} \\ & \times \prod_{j=1}^r \frac{(-4\pi i)^{\mathtt{k}_j} \sqrt{\sigma_j(m_1 m_2)}^{\mathtt{k}_j - 1}}{(\mathtt{k}_j - 2)!} J_{\mathtt{k}_j - 1}(4\pi \frac{\sqrt{\sigma_j(\eta m_1 m_2)}}{|\sigma_j(c)|}). \end{aligned}$$

Once again, we cannot assume $c \in F^+$ so $\omega_{\text{fin}}(c)$ may be nontrivial. In fact

$$\omega_{\text{fin}}(c) = \prod_{j=1}^r \operatorname{sgn}(\sigma_j(c))^{\mathtt{k}_j}.$$

Summing over all μ amounts to summing over nonzero $c \in \mathfrak{N}/\pm$, and we obtain the following.

Theorem 7.2. *With notation as above, suppose F has narrow class number 1, $\mathtt{k} = (\mathtt{k}_1, \dots, \mathtt{k}_r)$ with all $\mathtt{k}_j > 2$, and $\mathcal{F}$ is an orthogonal basis for $A_{\mathtt{k}}(\mathfrak{N}, \omega)$ consisting of eigenfunctions for the Hecke operator $T_{\mathfrak{n}}$. Then for any $m_1, m_2 \in \mathfrak{d}_+^{-1}$,*

$$\begin{aligned} & \frac{e^{2\pi \operatorname{tr}_{\mathbf{Q}}^F(m_1+m_2)}}{d_F^{1/2} \mathbb{N}(\eta)^{1/2} \psi(\mathfrak{N})} \left[\prod_{j=1}^r \frac{(\mathtt{k}_j - 2)!}{(4\pi \sqrt{\sigma_j(m_1 m_2)})^{\mathtt{k}_j - 1}} \right] \sum_{\varphi \in \mathcal{F}} \frac{\lambda_{\mathfrak{n}}^{\varphi} W_{m_1}^{\varphi}(1) \overline{W_{m_2}^{\varphi}(1)}}{\|\varphi\|^2} \\ & = T(dm_1, dm_2, \eta) \left[\prod_{j=1}^r (\operatorname{sgn} \sigma_j(\sqrt{m_1 m_2/\eta}))^{\mathtt{k}_j} \right] \omega_{\mathfrak{N}}(\sqrt{m_1 \eta/m_2})^{-1} \\ & + \frac{1}{d_F^{1/2}} \sum_{\substack{c \in N\mathcal{O}/\pm \\ c \neq 0}} S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta; c) \frac{(2\pi)^r}{\mathbb{N}(c)} \prod_{j=1}^r \frac{J_{\mathtt{k}_j - 1}(4\pi \frac{\sqrt{\sigma_j(\eta m_1 m_2)}}{|\sigma_j(c)|})}{(i \operatorname{sgn}(\sigma_j(c))^{\mathtt{k}_j}}. \end{aligned}$$

Remarks: (1) Because $\widehat{\mathcal{O}}/c\widehat{\mathcal{O}} \cong \mathcal{O}/c\mathcal{O}$, we see easily that for $c \in \mathfrak{N}$,

$$S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta; c) = \sum_{\substack{t_1, t_2 \in \mathcal{O}/c\mathcal{O} \\ t_1 t_2 \equiv \eta \bmod c\mathcal{O}}} \omega_{\mathfrak{N}}(t_2)^{-1} \theta_{\text{fin}}(\frac{m_1 t_1 + m_2 t_2}{c}).$$

The formula can be further simplified if we replace $c \in \mathfrak{N}$ by $c = Nu\tau$ for $u \in \mathcal{O}^*/\{\pm 1\}$ and $\tau \in \mathcal{O}/\mathcal{O}^*$. Then substituting $t_1' = u^{-1} t_1$, $t_2' = u^{-1} t_2$, we have

$$S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta; c) = \omega_{\mathfrak{N}}(u)^{-1} S_{\omega_{\mathfrak{N}}}(m_1, m_2; u^{-2}\eta; N\tau),$$

so we can break the sum over c into sums over u and τ and group together the terms with the same τ.

(2) If $F = \mathbf{Q}$, then $r = 1$. The sum over $c \in N\mathbf{Z}/\pm$ is simply a sum over $c > 0, N|c$, and we recover the generalized Petersson trace formula of [KL1].

Now take $\eta = 1$ so $\mathfrak{n} = \mathcal{O}$. Then $\lambda_{\mathfrak{n}}^{\varphi} = 1$ for all φ. Because the narrow class number is 1, $A_{\mathtt{k}}(\mathfrak{N}, \omega)$ corresponds (isometrically) to a classical space $S_{\mathtt{k}}(\mathfrak{N}, \omega')$ of cusp forms for $\Gamma_0(\mathfrak{N})$ as in (13). Here $\omega'(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \omega'(d) = \omega_{\mathfrak{N}}(d)^{-1}$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N}) = \mathrm{SL}_2(F) \cap K_0(\mathfrak{N})$. Conversely, given a character $\omega' : \Gamma_0(\mathfrak{N})/\Gamma_1(\mathfrak{N}) \to \mathbf{C}^*$, the space $S_{\mathtt{k}}(\mathfrak{N}, \omega')$ is isometric to $A_{\mathtt{k}}(\mathfrak{N}, \omega)$, where ω is the Hecke character determined by ω' using strong approximation $\mathbf{A}^* = F^*(F_\infty^+ \times \widehat{\mathcal{O}}^*)$ resulting from narrow class number 1. By (14), if $\varphi \leftrightarrow h$, then $W_m^{\varphi}(1) = d_F^{1/2} e^{-2\pi \operatorname{tr}(m)} a_m(h)$, and we obtain the following direct generalization of Petersson's original formula and (8) of [Lu].

Corollary 7.3. *For any orthogonal basis $\mathcal{F}$ for $S_{\mathtt{k}}(\mathfrak{N},\omega')$ and any $m_1, m_2 \in \mathfrak{d}_+^{-1}$ we have*

$$\frac{d_F^{1/2}}{\psi(\mathfrak{N})}\left[\prod_{j=1}^{r}\frac{(\mathtt{k}_j-2)!}{(4\pi\sqrt{\sigma_j(m_1m_2)})^{\mathtt{k}_j-1}}\right]\sum_{h\in\mathcal{F}}\frac{a_{m_1}(h)\overline{a_{m_2}(h)}}{\|h\|^2}$$

$$=\chi(m_1,m_2)+\frac{1}{d_F^{1/2}}\sum_{\substack{c\in N\mathcal{O}/\pm\\ c\neq 0}}S_{\omega'}(m_1,m_2;c)\frac{(2\pi)^r}{\mathbb{N}(c)}\prod_{j=1}^{r}\frac{J_{\mathtt{k}_j-1}(4\pi\frac{\sqrt{\sigma_j(m_1m_2)}}{|\sigma_j(c)|})}{(i\,\mathrm{sgn}(\sigma_j(c))^{\mathtt{k}_j}},$$

where $\chi(m_1,m_2)\in\{0,1\}$ is nonzero if and only if $m_2=m_1u^2$ for some $u\in\mathcal{O}^*$ and $S_{\omega'}(m_1,m_2;c)=\sum_{s\in(\mathcal{O}/c\mathcal{O})^*,\, s\bar{s}=1}\exp(2\pi i\,\mathrm{tr}_{\mathbf{Q}}^F(\frac{sm_1+\bar{s}m_2}{c}))\omega'(s)$.

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