WEIGHTED AVERAGES OF MODULAR L-VALUES

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Abstract. Using an explicit relative trace formula on GL(2), we derive a formula for averages of modular L-values in the critical strip, weighting by Fourier coefficients, Hecke eigenvalues, and Petersson norms. As an application we show that a GRH holds for these averages as the weight or the level goes to ∞. We also use the formula to give explicit zero-free regions of the form $|\operatorname{Im}(s)| \leq \tau_0$ for some particular modular L-functions.

July 30, 2009

1. Introduction

Let $S_{\mathtt{k}}(N, \omega')$ denote the space of cusp forms h on $\Gamma_0(N)$ satisfying

$$h(\frac{az+b}{cz+d}) = \omega'(d)^{-1}(cz+d)^{\mathtt{k}}\, h(z) \qquad (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)).$$

The Mellin transform of h is the analytic function

$$\Lambda(s,h) = \int_0^\infty h(iy) y^{s-1} dy,$$

which converges absolutely for all $s \in \mathbf{C}$ ([Sh], p. 94). Write $h(z) = \sum_{r>0} a_r(h) e^{2\pi i r z}$. When $\operatorname{Re}(s) > 1 + \mathtt{k}/2$, we have additionally

$$\int_0^\infty \sum_{r>0} |a_r(h) e^{-2\pi r y} y^{s-1}| dy < \infty.$$

Therefore for such s,

$$\Lambda(s,h) = \sum_{r>0} a_r(h) \int_0^\infty e^{-2\pi r y} y^{s-1} dy = \sum_{r>0} a_r(h) \int_0^\infty e^{-t} t^{s-1} (2\pi r)^{-s} dt$$
$$= (2\pi)^{-s}\Gamma(s) \sum_{r>0} \frac{a_r(h)}{r^s} = (2\pi)^{-s}\Gamma(s) L(s,h),$$

where $L(s,h)$ is the Dirichlet series attached to h. The completed L-function $\Lambda(s,h)$ satisfies a functional equation relating values at s and $\mathtt{k}-s$, which in the case of $N=1$ is simply

$$\Lambda(s,h) = i^{\mathtt{k}} \Lambda(\mathtt{k}-s, h). \tag{1}$$

Hence the critical line of the L-function is $\operatorname{Re}(s) = \mathtt{k}/2$. If h is a newform determining the cuspidal representation π, then $\Lambda(s,\pi) = \Lambda(s + \frac{\mathtt{k}-1}{2}, h)$, and $\Lambda(s,\pi)$ satisfies a functional equation relating its values at s and $1-s$.

The central values of L-functions have deep arithmetic significance. If the Hecke eigenvalues are known, one can compute the central values of a particular L-function using the approximate functional equation (see e.g. [Mi], §1.3.2). We can also use the trace formula to get information about averages of L-values as h ranges through an orthogonal Hecke eigenbasis $\mathcal{F}$ for $S_{\mathtt{k}}(N,\omega')$. In this paper, we will explicitly compute such an average, with the L-values weighted by Hecke eigenvalues, Fourier coefficients and Petersson norms.

The asymptotics of such averages have been studied widely. Duke showed that when $\mathtt{k}=2$, N is prime, ω' is trivial, and χ is a Dirichlet character unramified at N,

$$\frac{1}{\psi(N)}\sum_{h\in\mathcal{F}}\frac{a_1(h)L(1,h\otimes\chi)}{\|h\|^2}=4\pi+O(N^{-1/2}\log N),$$

where $\psi(N)=[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(N)]$, [Du]. Here we have normalized the Petersson norm as in (2) below. With a more careful estimation, Ellenberg improved Duke's error term to $O(N^{-1+\varepsilon})$, while at the same time allowing $a_r(h)$ in place of $a_1(h)$, [El]. Of the many other generalizations of Duke's work, we mention two: Akbary extended it to weight $\mathtt{k}>2$ with an error term of $O_{\mathtt{k}}(N^{-1/2}(\log N)^{\mathtt{k}-1})$ [Ak], and Kamiya further allowed composite N and $L(1+it,h\otimes\chi)$ with an error term of $O_{t,\mathtt{k}}(N^{-\mathtt{k}/4})$ [Ka]. The method of Duke uses the Petersson trace formula. Another approach, based on the Eichler-Selberg trace formula, was found by Royer (see §4.3 of [Ro]).

Here we consider the case $\mathtt{k}>2$. For the weighted averages we obtain an error term of $O(N^{-\mathtt{k}/2})$ on the critical line. In fact, we give an explicit formula for the average (Theorem 1.1). At the same time, we allow s to vary through the whole critical strip. We will also give the asymptotic behavior of the average as $\mathtt{k}\to\infty$.

To state the main theorem, for $h\in S_{\mathtt{k}}(N,\omega')$, let $h^-\in S_{\mathtt{k}}(N,\omega'^{-1})$ denote the "complex conjugate" of h, given by $h^-(z)=\sum\overline{a_n(h)}q^n$. If ω' is trivial, then $h^-=h$, and in general $\Lambda(s,h^-)=\overline{\Lambda(\bar{s},h)}$.

Theorem 1.1. *Let $r,N,\mathtt{n},\mathtt{k}\in\mathbf{Z}^+$ with $(\mathtt{n},N)=1$ and $\mathtt{k}>2$. Fix a Dirichlet character ω' of conductor dividing N, and suppose $S_{\mathtt{k}}(N,\omega')\neq\{0\}$. Let $\mathcal{F}$ be an orthogonal basis for $S_{\mathtt{k}}(N,\omega')$ consisting of eigenfunctions for the Hecke operator $T_{\mathtt{n}}$. Then for any $s\in\mathbf{C}$ with $1<\mathrm{Re}(s)<\mathtt{k}-1$,*

$$\begin{aligned}\sum_{h\in\mathcal{F}}&\frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s,h^-)}{\|h\|^2}\\ =&\frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!}\sum_{m|\gcd(\mathtt{n},r)}\frac{m^{2s-\mathtt{k}+1}}{\omega'(m)}\\ &+\delta_{N,1}\frac{2^{\mathtt{k}-1}\Gamma(\mathtt{k}-s)(2\pi r\mathtt{n})^{s-1}}{(\mathtt{k}-2)!\,i^{\mathtt{k}}}\sum_{m|\gcd(\mathtt{n},r)}m^{\mathtt{k}-2s+1}\\ &+\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\,e^{i\pi s/2}}\sum_{\substack{a\neq0,d>0\\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}}\frac{a^{-(\mathtt{k}-s)}d^{-s}\gcd(a,Nd)}{\omega'(a)\,e^{2\pi ir\ell_0/a}}\,{}_1f_1(s;\mathtt{k};\tfrac{2\pi ir\mathtt{n}}{Nad}),\end{aligned}$$

where $T_{\mathtt{n}}h=\lambda_{\mathtt{n}}(h)h$, ℓ_0 is any integer satisfying $\ell_0Nd\equiv\mathtt{n}\bmod a$, and

$${}_1f_1(s;\mathtt{k};w)=\frac{\Gamma(s)\Gamma(\mathtt{k}-s)}{\Gamma(\mathtt{k})}{}_1\mathrm{F}_1(s;\mathtt{k};w)$$

for the confluent hypergeometric function ${}_1\mathrm{F}_1(s;\mathtt{k};w)=1+\frac{s}{\mathtt{k}}w+\frac{s(s+1)}{\mathtt{k}(\mathtt{k}+1)}\frac{w^2}{2!}+\cdots$. When $a<0$, we take $a^s=e^{i\pi s}|a|^s$. Throughout we use the convention that $\sum_{m|n}$ is a sum over positive *divisors of n.*

This theorem generalizes a result of Kohnen, who derived the special case $\mathtt{n} = N = 1$ using a Poincaré series-type argument ([Ko], p. 188). Our approach here is quite different.

From its integral representation (cf. (17) on page 16), it follows that

$$|{}_1f_1(s;\mathtt{k};2\pi i r\mathtt{n}/Nad)| \leq 1.$$

Thus the sum over a, d is bounded independently of N (see Prop. 4.2 for a precise bound), and we have the following.

Corollary 1.2. *With notation as above and* $1 < \mathrm{Re}(s) < \mathtt{k} - 1$*,*

$$\frac{1}{\psi(N)} \sum_{h\in\mathcal{F}} \frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s,h^-)}{\|h\|^2} \\ = \frac{2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!} \sum_{m|\,\gcd(\mathtt{n},r)} \frac{m^{2s-\mathtt{k}+1}}{\omega'(m)} + O(N^{-\mathrm{Re}(s)}).$$

The implied constant is effective and depends only on $\mathtt{k}, \mathtt{n}, r$ *and* s*, uniformly for* s *in compact subsets of the given strip.*

According to the Grand Riemann Hypothesis, when h is a Hecke eigenform all zeros of $\Lambda(s,h)$ inside the critical strip $\frac{\mathtt{k}-1}{2} < \mathrm{Re}(s) < \frac{\mathtt{k}+1}{2}$ lie on the critical line $\mathrm{Re}(s) = \mathtt{k}/2$. Using Theorem 1.1, we will show that a GRH holds for averages (see also [Ko] for the $N = 1$ case). Note that Corollary 1.2 implies nonvanishing of the average when N is large, at least when $\gcd(\mathtt{n}, r) = 1$. By the results of Section 4.1 in which we determine the asymptotic behavior as $\mathtt{k} \to \infty$, the average is also nonzero when $\mathtt{k}$ is large. To state the result, we shift the L-functions so that the critical strip becomes $0 \leq \mathrm{Re}(s) \leq 1$, independent of $\mathtt{k}$.

Corollary 1.3. *Assume* $N > 1$*,* $\mathtt{k} > 3$*,* $\gcd(\mathtt{n}, r) = 1$*, and that* $S_{\mathtt{k}}(N, \omega') \neq \{0\}$*. For* $\tau_0 > 0$*, let* R *be the rectangle consisting of* s *with* $0 \leq \mathrm{Re}(s) \leq 1$ *and* $|\mathrm{Im}(s)| \leq \tau_0$*. Then there exist constants* $C_{\mathtt{k}}, C_N > 0$ *depending only on* R*,* $\mathtt{n}$ *and* r*, such that if either* $\mathtt{k} > C_{\mathtt{k}}$ *or* $N > C_N$*, the sum*

$$\sum_{h\in\mathcal{F}} \frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s+\frac{\mathtt{k}-1}{2},h^-)}{\|h\|^2}$$

is nonzero for every $s \in R$*. In particular, for any* $s \in R$ *there exists an eigenform* $h \in S_{\mathtt{k}}(N, \omega')$ *such that* $\lambda_{\mathtt{n}}(h), a_r(h)$ *and* $\Lambda(s + \frac{\mathtt{k}-1}{2}, h)$ *are all nonzero.*

Some of the hypotheses of Corollary 1.3 can be weakened with minor modifications. To allow $\gcd(\mathtt{n}, r) > 1$, we simply need to exclude the left edge of the strip. Thus the boundary of R should be shrunk to $\delta \leq \mathrm{Re}(s) \leq 1$ for any $0 < \delta < 1/2$. If in addition we exclude the right edge by considering $\delta \leq \mathrm{Re}(s) \leq 1 - \delta$ for such δ, then the statement is also valid for $\mathtt{k} = 3$. When $N = 1$, the situation is a little more delicate because, if s lies on the critical line, the first two terms in the formula for the average may cancel each other out and we cannot say anything. Indeed if $\mathtt{k} \equiv 2 \bmod 4$, the L-values themselves vanish at $s = \mathtt{k}/2$ because of the functional equation (1). So when $N = 1$ we must assume that R is a compact region which does not meet the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Suppose it happens that $\dim S_{\mathtt{k}}(N, \omega') = 1$. Then the theorem gives a computable formula for the values of the L-function of the cusp form. Using an effective version of Corollary 1.3, we obtain zero-free regions for several such L-functions in

Section 4.2. As a final illustration, we show how to use the formula to compute some familiar data, namely values of Ramanujan's τ-function. This is achieved by taking a quotient of two different averages. The resulting expression can be estimated to any desired precision using partial sum approximations, and since $\tau(r)$ is known to be an integer, we can pinpoint its value with just a few terms.

Theorem 1.1 is proven using a relative trace formula on GL(2). We start with a Hecke operator and integrate its associated kernel over the group $N \times M$, where N is unipotent and M is diagonal. This is a hybrid of the techniques of the papers [Li], [KL1] (which used $N \times N$) and [RaRo] (which used $M \times M$). The paper [RaRo] of Ramakrishnan and Rogawski gives an asymptotic formula for certain averages of the form $\sum_{h\in\mathcal{F}} \frac{\lambda_{p^n}(h)\Lambda(\mathtt{k}/2,h\otimes\chi)\Lambda(\mathtt{k}/2,h)}{\|h\|^2}$, which yields a weighted equidistribution result for the Hecke eigenvalues. They use a regularization procedure since they assert that the terms on their geometric side are not absolutely convergent. Thus the replacement here of just one factor of M by the unipotent group N (of compact quotient) is enough to give an absolutely convergent trace formula.

We mention that Feigon and Whitehouse refined the method of [RaRo] in many cases by using the Jacquet-Langlands correspondence to avoid the convergence issues inherent to GL(2), [FW]. They obtain closed formulas for the averages at the central point, over a totally real field.

A version of Theorem 1.1 involving twisted L-functions $\Lambda(s, h \otimes \chi)$ should be obtainable by similar methods, using a test function as in [RaRo]. Of course, the presence of a nontrivial character χ will only help the convergence of the trace formula.

We would like to thank David Bradley and George Knightly for their helpful comments on the hypergeometric function. The numerical calculations in Section 4 were made using Mathematica. The first author was supported by the University of Maine Office of the Vice President for Research, and NSA grant H98230-06-1-0039.

2. Notation and preliminaries

We briefly recall the notation and test function of [KL2], which contains proofs of the various facts mentioned in this section. Let $\mathbf{A}, \mathbf{A}_{\text{fin}}$ be the adeles and finite adeles of $\mathbf{Q}$, and let $G = \mathrm{GL}(2)$. We write $\overline{G}$ for G/Z, where Z is the center. Fix a level $N \geq 1$ and a Dirichlet character ω' of conductor dividing N. For a weight $\mathtt{k} > 2$, let $S_{\mathtt{k}}(N, \omega')$ denote the space of cusp forms satisfying

$$h(\gamma z) = \omega'(\gamma)^{-1} j(\gamma, z)^{\mathtt{k}} h(z) \qquad (\gamma \in \Gamma_0(N)).$$

Here $\omega'(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \omega'(d)$ and

$$j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (ad - bc)^{-1/2}(cz + d) \qquad (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{R})^+).$$

Using $\mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}_+^* \times \widehat{\mathbf{Z}}^*)$, define

$$\omega : \mathbf{A}^* \to \widehat{\mathbf{Z}}^* \to (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*,$$

where the last arrow is ω'. For an idele x, let x_N denote the idele which agrees with x at the places $p|N$, and which is 1 at all other places. Then for any integer d prime to N,

$$\omega(d_N) = \omega'(d).$$

To each $h \in S_{\mathtt{k}}(N, \omega')$ we associate $\phi_h \in L_0^2(\omega) = L_0^2(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}), \omega)$ by using strong approximation:

$$\phi_h(\gamma(g_\infty \times k)) = j(g_\infty, i)^{-\mathtt{k}} h(g_\infty(i))$$

for $\gamma \in G(\mathbf{Q}), g_\infty \in G(\mathbf{R})^+$ and $k \in K_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbf{Z}}) |\, c, d-1 \in N\widehat{\mathbf{Z}}\}$.

We normalize the Petersson norm by

$$\|h\|^2 = \frac{1}{\psi(N)} \int_{\Gamma_0(N)\backslash\mathbf{H}} |h(z)|^2 y^{\mathtt{k}} \frac{dx\, dy}{y^2}. \tag{2}$$

If we normalize Haar measure on $\overline{G}(\mathbf{A})$ so that $\mathrm{meas}(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})) = \pi/3$, then the Petersson norm corresponds to the L^2-norm and the map $h \mapsto \phi_h$ is an isometry. We normalize Haar measure on $\mathbf{A}$ so that $\mathrm{meas}(\mathbf{Q}\backslash\mathbf{A}) = 1$. We take Lebesgue measure dx on $\mathbf{R}$ and $d^*y = \frac{dy}{|y|}$ on $\mathbf{R}^*$. On $\mathbf{A}_{\mathrm{fin}}^*$ we normalize so that $\mathrm{meas}(\widehat{\mathbf{Z}}^*) = 1$.

Fix $\mathtt{n} \in \mathbf{Z}^+$ with $\gcd(\mathtt{n}, N) = 1$, and define a test function $f = f_\infty \times f^{\mathtt{n}}$ as follows. Define

$$M(\mathtt{n}, N) = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\widehat{\mathbf{Z}}) |\, \det g \in \mathtt{n}\widehat{\mathbf{Z}}^* \text{ and } c \equiv 0 \bmod N\widehat{\mathbf{Z}}\}.$$

The support of $f_{\mathrm{fin}} = f^{\mathtt{n}}$ is the set $Z(\mathbf{A}_{\mathrm{fin}})M(\mathtt{n}, N) = Z(\mathbf{Q}^+)M(\mathtt{n}, N)$. By definition,

$$f^{\mathtt{n}}(z_{\mathbf{Q}} m) = \frac{\psi(N)}{\omega(m)} \qquad (z_{\mathbf{Q}} \in Z(\mathbf{Q}^+),\, m \in M(\mathtt{n}, N)),$$

where for $m = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(\mathtt{n}, N)$ we define $\omega(m) = \omega(d_N)$. We take $f_\infty(g) = \frac{1}{d_{\mathtt{k}}}\overline{\langle \pi_{\mathtt{k}}(g)v_0, v_0 \rangle}$, where $\pi_{\mathtt{k}}$ is the weight $\mathtt{k}$ discrete series of $\mathrm{GL}_2(\mathbf{R})$ with formal degree $d_{\mathtt{k}} = \frac{4\pi}{\mathtt{k}-1}$ and lowest weight unit vector v_0. Explicitly, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$f_\infty(g) = \begin{cases} \frac{(\mathtt{k}-1)}{4\pi} \frac{\det(g)^{\mathtt{k}/2}(2i)^{\mathtt{k}}}{(-b+c+(a+d)i)^{\mathtt{k}}} & \text{if } \det(g) > 0, \\ 0 & \text{otherwise} \end{cases}$$

(see [KL2], Theorem 14.5). By construction, $f(zg) = \omega(z)^{-1} f(g)$ for $z \in Z(\mathbf{A})$.

This function f is integrable precisely when $\mathtt{k} > 2$. Hence for such $\mathtt{k}$ it defines an operator $R(f)$ on $L^2(\omega)$ by

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(g)\phi(xg) dg.$$

Then as shown in [KL2], we have the following commutative diagram:

$$\begin{array}{ccc} L^2(\omega) & \xrightarrow{\ \mathtt{n}^{\frac{\mathtt{k}}{2}-1} R(f)\ } & L^2(\omega) \\ {\scriptstyle \text{orthog. proj.}} \downarrow & & \uparrow \\ S_{\mathtt{k}}(N, \omega') & \xrightarrow{\ T_{\mathtt{n}}\ } & S_{\mathtt{k}}(N, \omega') \end{array}$$

where $T_{\mathtt{n}}$ is the classical Hecke operator. Letting $\mathcal{F}$ be any orthogonal basis for $S_{\mathtt{k}}(N, \omega')$, the kernel of $R(f)$ is the function on $G(\mathbf{A}) \times G(\mathbf{A})$ given by

$$K(g_1, g_2) = \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(g_1^{-1}\gamma g_2) = \sum_{h \in \mathcal{F}} \frac{R(f)\phi_h(g_1)\overline{\phi_h(g_2)}}{\|\phi_h\|^2}. \tag{3}$$

Lastly, we let $\theta : \mathbf{A} \to \mathbf{C}^*$ denote the standard character of $\mathbf{A}$. It is defined by

$$\theta_\infty(x) = e^{-2\pi i x}, \quad x \in \mathbf{R},$$

and

$$\theta_p(x) = e^{2\pi i r_p(x)}, \quad x \in \mathbf{Q}_p,$$

where $r_p(x) \in \mathbf{Q}$ is the principal part of x, a number with p-power denominator characterized (up to $\mathbf{Z}_p$) by $x \in r_p(x) + \mathbf{Z}_p$. Then θ is trivial on $\mathbf{Q}$ and $\theta_{\text{fin}} = \prod_p \theta_p$ is trivial precisely on $\widehat{\mathbf{Z}}$. In particular, for any $q \in \mathbf{Q}$, $\theta_{\text{fin}}(q) = \theta_\infty(q)^{-1} = e^{2\pi i q}$. The characters of $\mathbf{Q}\backslash\mathbf{A}$ are parametrized by $r \in \mathbf{Q}$ via

$$\theta_r(x) = \theta(-rx).$$

3. Proof of the theorem

3.1. **Spectral side.** The theorem is proven by computing the following

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \int_{\mathbf{Q}\backslash\mathbf{A}} K(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) \overline{\theta_r(x)} |y|^{s-\mathtt{k}/2} dx\, d^*y \tag{4}$$

using the two expressions for the kernel (3). We will see presently that the integral (4) is absolutely convergent for all s.

For the spectral side, choose $\mathcal{F}$ in (3) to be an orthogonal basis of eigenvectors of $T_{\mathtt{n}}$. Then $R(f)\phi_h = \mathtt{n}^{1-\mathtt{k}/2}\lambda_{\mathtt{n}}(h)\phi_h$ for $h \in \mathcal{F}$, so (4) is equal to

$$\sum_{h\in\mathcal{F}} \frac{\mathtt{n}^{1-\mathtt{k}/2}\lambda_{\mathtt{n}}(h)}{\|\phi_h\|^2} \int_{\mathbf{Q}\backslash\mathbf{A}} \phi_h(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))\overline{\theta_r(x)} dx \int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \overline{\phi_h(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))} |y|^{s-\mathtt{k}/2} d^*y \tag{5}$$

$$= \frac{\mathtt{n}^{1-\mathtt{k}/2}}{e^{2\pi r}} \sum_{h\in\mathcal{F}} \frac{\lambda_{\mathtt{n}}(h) a_r(h) \Lambda(s, h^-)}{\|h\|^2},$$

by the following lemma.

Lemma 3.1. *For $r \in \mathbf{Q}$,*

$$\int_{\mathbf{Q}\backslash\mathbf{A}} \phi_h(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))\, \overline{\theta_r(x)} dx = \begin{cases} e^{-2\pi r} a_r(h) & \text{if } r \in \mathbf{Z}^+, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \overline{\phi_h(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))}\, |y|^{s-\mathtt{k}/2} d^*y = \Lambda(s, h^-).$$

Proof. For a proof of the first statement, see [KL2], Corollary 12.4. For the second, note that $\phi_h(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)) = y^{\mathtt{k}/2} h(iy)$ when $y \in \mathbf{R}_+^*$. Furthermore, $\overline{h(iy)} = \sum \overline{a_r(h)} e^{-2\pi r y} = h^-(iy)$. We can integrate over the fundamental domain $\mathbf{R}_+^* \times \widehat{\mathbf{Z}}^*$. The integrand is invariant under $\widehat{\mathbf{Z}}^*$, which has measure 1. Thus

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \overline{\phi_h(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))} |y|^{s-\mathtt{k}/2} d^*y = \int_0^\infty h^-(iy) y^{s-1} dy = \Lambda(s, h^-).$$

□

The two integrals in (5) are absolutely convergent for all s, so we have the following.

Proposition 3.2. *The double integral* (4) *is absolutely convergent for all $s \in \mathbf{C}$.*

3.2. **Geometric side.** On the geometric side, we use the formalism of Jacquet's relative trace formula. Let N be the upper triangular unipotent subgroup of G, and let M be the diagonal subgroup. Let $\overline{M} = M/Z$, where Z is the center. Setting $H = N \times \overline{M}$, the integral (4) is taken over $H(\mathbf{Q})\backslash H(\mathbf{A})$. Using $K(n,m) = \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(n^{-1}\gamma m)$, we would like to pull the sum out of (4); however the individual terms $f(n^{-1}\gamma m)$ are not well-defined modulo $H(\mathbf{Q})$. We have to break $\overline{G}(\mathbf{Q})$ into $H(\mathbf{Q})$-orbits and then sum over these orbits. The action of H is $(n,m)\cdot\gamma = n^{-1}\gamma m$. For $\delta \in \overline{G}(\mathbf{Q})$, its orbit is

$$[\delta] = \{\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right)\delta\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \,|\, x \in \mathbf{Q}, y \in \mathbf{Q}^*\} = \{n^{-1}\delta m \,|\, (n,m) \in H_\delta(\mathbf{Q})\backslash H(\mathbf{Q})\},$$

where H_δ is the stabilizer of δ. It is easy to check that in fact $H_\delta = \{1\}$ for any δ. Thus the geometric expression for (4) is equal to

$$\sum_{[\delta]} \int_{\mathbf{A}^*} \int_{\mathbf{A}} f(\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right)\delta\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))\overline{\theta_r(x)}|y|^{s-\mathtt{k}/2} dx\, d^*y. \tag{6}$$

To justify this manipulation we have to show that (6) converges absolutely.

Proposition 3.3. *Suppose* $1 < \mathrm{Re}(s) < \mathtt{k}-1$. *Then*

$$\sum_{[\delta]} \int_{\mathbf{A}^*} \int_{\mathbf{A}} \left| f(\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right)\delta\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))\overline{\theta_r(x)}|y|^{s-\mathtt{k}/2} \right| dx\, d^*y < \infty.$$

Thus for such s, the geometric side (6) *converges absolutely and equals the spectral side* (5).

We postpone the proof of the proposition until Section 3.3 below. Assuming it for now, let $I_\delta(f)$ denote the double integral attached to δ in (6). By the proposition, $I_\delta(f)$ is absolutely convergent on the given strip. We just need to determine the set of δ and compute each of these geometric integrals. We assume throughout that the hypothesis of the proposition is satisfied.

The set of orbits $[\delta]$ is in one-to-one correspondence with $N(\mathbf{Q})\backslash\overline{G}(\mathbf{Q})/\overline{M}(\mathbf{Q})$. By the Bruhat decomposition

$$G(\mathbf{Q}) = N(\mathbf{Q})M(\mathbf{Q}) \cup N(\mathbf{Q})\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)N(\mathbf{Q})M(\mathbf{Q}),$$

a set of representatives is given by

$$\{1\} \cup \{\left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right) \,|\, t \in \mathbf{Q}\}.$$

Proposition 3.4. *When $\delta = 1$, the integral*

$$I_1(f) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta(rx) dx |y|^{s-\mathtt{k}/2} d^*y$$

converges absolutely on $0 < \mathrm{Re}(s) < \mathtt{k}-1$, *and for such s it is*

$$= \frac{\mathtt{n}^{1-\mathtt{k}/2}}{e^{2\pi r}} \frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!} \sum_{m \mid \gcd(\mathtt{n},r)} \frac{m^{2s-\mathtt{k}+1}}{\omega'(m)}.$$

Proof. The absolute convergence will be proven in Prop. 3.10 below. For s as given, we factorize the integral as $I_1(f)_\infty I_1(f)_{\mathrm{fin}}$. To start with,

$$I_1(f)_{\mathrm{fin}} = \int_{\mathbf{A}^*_{\mathrm{fin}}} \int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta_{\mathrm{fin}}(rx) dx |y|_{\mathrm{fin}}^{s-\mathtt{k}/2} d^*y.$$

The value of $f^{\mathtt{n}}$ is nonzero if and only if there exists $m \in \mathbf{Q}^+$ such that $\left(\begin{smallmatrix} my & -mx \\ 0 & m \end{smallmatrix}\right) \in M(\mathtt{n}, N)$. In particular, $m \in \widehat{\mathbf{Z}} \cap \mathbf{Q}^+ = \mathbf{Z}^+$. Furthermore,

(i) $my \in \widehat{\mathbf{Z}}$
(ii) $m^2 y \in \mathtt{n}\widehat{\mathbf{Z}}^*$
(iii) $mx \in \widehat{\mathbf{Z}}$.

Together, the first two conditions imply that $m|\mathtt{n}$. Conversely, if $m|\mathtt{n}$, condition (ii) implies condition (i). Assuming that $m|\mathtt{n}$ and y satisfies (ii), we have

$$\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\left(\begin{smallmatrix} y & -x \\ 0 & 1 \end{smallmatrix}\right))\theta_{\mathrm{fin}}(rx)dx = \frac{\psi(N)}{\omega(m_N)} \int_{\frac{1}{m}\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(rx)dx.$$

Because $m|\mathtt{n}$, it follows that $(m, N) = 1$, so $\omega(m_N) = \omega'(m)$. Hence the above is

$$= \begin{cases} m\psi(N)/\omega'(m) & \text{if } m|r \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta_{\mathrm{fin}}(rx)dx = \begin{cases} m\psi(N)/\omega'(m) & \text{if } y \in \frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^* \text{ for} \\ & \text{ some } m|\gcd(\mathtt{n}, r), \\ 0 & \text{otherwise.} \end{cases}$$

We note that if such m exists, it is uniquely determined by y. Now

$$I_1(f)_{\mathrm{fin}} = \sum_{m|\gcd(\mathtt{n},r)} \frac{m\psi(N)}{\omega'(m)} \int_{\frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^*} |y|_{\mathrm{fin}}^{s-\mathtt{k}/2} d^*y = \psi(N) \sum_{m|\gcd(\mathtt{n},r)} \frac{m(m^2/\mathtt{n})^{s-\mathtt{k}/2}}{\omega'(m)}.$$

For the infinite part, recall that f_∞ vanishes on matrices with negative determinant. Thus

$$I_1(f)_\infty = \int_0^\infty \int_{\mathbf{R}} f_\infty(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta_\infty(rx)dx\, |y|^{s-\mathtt{k}/2} d^*y.$$

We have

$$\int_{\mathbf{R}} f_\infty(\left(\begin{smallmatrix} y & -x \\ 0 & 1 \end{smallmatrix}\right))\theta_\infty(rx)dx = \frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \int_{-\infty}^{\infty} \frac{e^{-2\pi irx}}{(x+(y+1)i)^{\mathtt{k}}} dx.$$

Use a clockwise semicircular contour integral in the lower complex half-plane. The integrand has a pole at $x = -(y+1)i$ inside the contour. By the residue theorem, the above is

$$= -\frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \frac{2\pi i}{(\mathtt{k}-1)!} \left.\frac{d^{\mathtt{k}-1}}{dx^{\mathtt{k}-1}}\right|_{x=-(y+1)i} e^{-2\pi irx}$$

$$= -\frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \frac{2\pi i}{(\mathtt{k}-1)!} (-2\pi ir)^{\mathtt{k}-1} e^{-2\pi r(y+1)} = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)!\, e^{2\pi r}}\, y^{\mathtt{k}/2} e^{-2\pi ry}.$$

Therefore using $\mathrm{Re}(s) > 0$,

$$I_1(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)! e^{2\pi r}} \int_0^\infty y^{s-1} e^{-2\pi ry} dy = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)! e^{2\pi r}} (2\pi r)^{-s}\Gamma(s).$$

All together we have

$$I_1(f) = \frac{\psi(N) 2^{\mathtt{k}-1}\Gamma(s)\mathtt{n}^{\mathtt{k}/2-s}(2\pi r)^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!\, e^{2\pi r}} \sum_{m|\gcd(\mathtt{n},r)} \frac{m^{2s-\mathtt{k}+1}}{\omega'(m)}.$$

□

Next we need to compute $I_\delta(f)$ for $\delta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right)$ with $t \in \mathbf{Q}$. We begin with the special case $t = 0$.

Proposition 3.5. *If* $\delta = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, *then*

$$I_\delta(f) = \int_{\mathbf{A}^*}\int_{\mathbf{A}} f(\begin{pmatrix} yx & 1 \\ -y & 0 \end{pmatrix})\,\theta(rx)dx|y|^{s-\mathtt{k}/2}d^*y \tag{7}$$

converges absolutely for $1 < \mathrm{Re}(s) < \mathtt{k}$. *For such* s, $I_\delta(f) = 0$ *unless* $N = 1$. *When* $N = 1$,

$$I_\delta(f) = \frac{\mathtt{n}^{1-\mathtt{k}/2}}{e^{2\pi r}} \frac{2^{\mathtt{k}-1}\Gamma(\mathtt{k}-s)(2\pi r\mathtt{n})^{s-1}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}} \sum_{m|\gcd(\mathtt{n},r)} m^{\mathtt{k}-2s+1}.$$

Proof. For the absolute convergence, see Prop. 3.10 below. The value of $f^{\mathtt{n}}$ in $I_\delta(f)_{\mathrm{fin}} = \int_{\mathbf{A}^*_{\mathrm{fin}}}\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1 \\ -y & 0 \end{smallmatrix}\right))\theta_{\mathrm{fin}}(rx)dx|y|^{s-\mathtt{k}/2}_{\mathrm{fin}}d^*y$ is nonzero if and only if there exists $m \in \mathbf{Q}^+$ such that $\left(\begin{smallmatrix} myx & m \\ -my & 0 \end{smallmatrix}\right) \in M(\mathtt{n}, N)$. This means $m \in \mathbf{Z}^+$, $my \in N\widehat{\mathbf{Z}}$ and $m^2y \in \mathtt{n}\widehat{\mathbf{Z}}^*$. It follows that $N|\mathtt{n}$, which is only possible if $N = 1$. Assuming $N = 1$, we have $m|\mathtt{n}$. The last requirement for nonvanishing is $x \in \frac{1}{my}\widehat{\mathbf{Z}} = \frac{m}{\mathtt{n}}\widehat{\mathbf{Z}}$, in which case $f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1 \\ -y & 0 \end{smallmatrix}\right)) = 1$. Hence for fixed $m|\mathtt{n}$ and $y \in \frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^*$,

$$\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} yx & 1 \\ -y & 0 \end{pmatrix})\,\theta_{\mathrm{fin}}(rx)dx = \int_{\frac{m}{\mathtt{n}}\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(rx)dx = \begin{cases} \mathtt{n}/m & \text{if } \frac{rm}{\mathtt{n}} \in \widehat{\mathbf{Z}} \\ 0 & \text{otherwise.} \end{cases}$$

Now we have

$$\begin{aligned} I_\delta(f)_{\mathrm{fin}} &= \sum_{\substack{m|\mathtt{n},\\ \frac{\mathtt{n}}{m}|r}} \frac{\mathtt{n}}{m}\int_{\frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^*} |y|^{s-\mathtt{k}/2}_{\mathrm{fin}}d^*y = \sum_{\substack{m|\mathtt{n},\\ \frac{\mathtt{n}}{m}|r}} \frac{\mathtt{n}}{m}(m^2/\mathtt{n})^{s-\mathtt{k}/2} \\ &= \mathtt{n}^{\mathtt{k}/2-s+1}\sum_{\substack{m|\mathtt{n},\\ \frac{\mathtt{n}}{m}|r}} m^{2s-\mathtt{k}-1} = \mathtt{n}^{\mathtt{k}/2-s+1}\sum_{m|\gcd(\mathtt{n},r)} (\mathtt{n}/m)^{2s-\mathtt{k}-1} \\ &= \mathtt{n}^{s-\mathtt{k}/2}\sum_{m|\gcd(\mathtt{n},r)} m^{\mathtt{k}-2s+1}. \end{aligned}$$

For the infinite part $I_\delta(f)_\infty = \int_{\mathbf{R}^*}\int_{\mathbf{R}} f_\infty(\left(\begin{smallmatrix} yx & 1 \\ -y & 0 \end{smallmatrix}\right))\theta_\infty(rx)dx|y|^{s-\mathtt{k}/2}d^*y$, as before we can assume $y > 0$. We have

$$\begin{aligned} \int_{\mathbf{R}} f_\infty(\begin{pmatrix} yx & 1 \\ -y & 0 \end{pmatrix})e^{-2\pi irx}dx &= \frac{\mathtt{k}-1}{4\pi}y^{\mathtt{k}/2}(2i)^{\mathtt{k}}\int_{-\infty}^{\infty}\frac{e^{-2\pi irx}}{(-1-y+(yx)i)^{\mathtt{k}}}dx \\ &= \frac{(\mathtt{k}-1)(2i)^{\mathtt{k}}}{4\pi}y^{\mathtt{k}/2}(iy)^{-\mathtt{k}}\int_{-\infty}^{\infty}\frac{e^{-2\pi irx}}{(x+(\frac{1+y}{y})i)^{\mathtt{k}}}dx. \end{aligned}$$

Take a clockwise semicircular contour integral in the lower half-plane. The integrand has a pole at $x = -i(1+\frac{1}{y})$. By the residue theorem the above is

$$\begin{aligned} &= -\frac{(\mathtt{k}-1)2^{\mathtt{k}}}{4\pi}y^{-\mathtt{k}/2}\frac{2\pi i}{(\mathtt{k}-1)!}\,\frac{d^{\mathtt{k}-1}}{dx^{\mathtt{k}-1}}\Big|_{x=-i(1+\frac{1}{y})} e^{-2\pi irx} \\ &= \frac{-i2^{\mathtt{k}-1}}{(\mathtt{k}-2)!}\,y^{-\mathtt{k}/2}(-2\pi ir)^{\mathtt{k}-1}e^{-2\pi r(1+1/y)} \\ &= \frac{(4\pi r)^{\mathtt{k}-1}e^{-2\pi r}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}}\,y^{-\mathtt{k}/2}e^{-2\pi r/y}. \end{aligned}$$

Therefore

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}e^{-2\pi r}}{(\mathtt{k}-2)!\,i^{\mathtt{k}}}\int_0^\infty y^{s-\mathtt{k}-1}e^{-2\pi r/y}dy.$$

For any $\alpha > 0$, $\int_0^\infty t^{w-1}e^{-\alpha/t}dt = \alpha^w\Gamma(-w)$ when $\mathrm{Re}(w) < 0$, so we get

$$I_\delta(f) = \frac{(4\pi r)^{\mathtt{k}-1}e^{-2\pi r}}{(\mathtt{k}-2)!\,i^{\mathtt{k}}}(2\pi r)^{s-\mathtt{k}}\Gamma(\mathtt{k}-s)\mathtt{n}^{s-\mathtt{k}/2}\sum_{m|\gcd(\mathtt{n},r)} m^{\mathtt{k}-2s+1}.$$

□

For the case of $\delta = \left(\begin{smallmatrix}0&1\\-1&t\end{smallmatrix}\right)$ with $t \in \mathbf{Q}^*$, we use the following lemma, which is very easy to prove.

Lemma 3.6. *For any $n, m, r \in \widehat{\mathbf{Z}}$,*

$$r\widehat{\mathbf{Z}} \cap (n + m\widehat{\mathbf{Z}}) = \begin{cases} rc_0 + \frac{rm}{\gcd(r,m)}\widehat{\mathbf{Z}} & \text{if } \gcd(r,m)|n \\ \emptyset & \text{if } \gcd(r,m)\nmid n, \end{cases}$$

where $c_0 \in \mathbf{Z}$ is any fixed solution to $rc_0 \equiv n \bmod m\widehat{\mathbf{Z}}$.

We also need to recall the definition of the confluent hypergeometric function

$${}_1\mathrm{F}_1(s;k;w) = \sum_{m=0}^\infty \frac{(s)_m}{(k)_m}\frac{w^m}{m!}$$

where $(s)_0 = 1$ and for $m > 0$, $(s)_m = s(s+1)(s+2)\cdots(s+m-1)$. This is absolutely convergent for all $s, k, w \in \mathbf{C}$, except when k is a nonpositive integer. We have the following useful integral representation:

$$(8)\quad {}_1\mathrm{F}_1(s;k;w) = \frac{\Gamma(k)}{\Gamma(k-s)\Gamma(s)}\int_0^1 e^{wt}t^{s-1}(1-t)^{k-s-1}dt \quad (\mathrm{Re}(k) > \mathrm{Re}(s) > 0)$$

(see [Sl], §3.1).

Proposition 3.7. *If $\delta = \left(\begin{smallmatrix}0&1\\-1&t\end{smallmatrix}\right)$ for $t \in \mathbf{Q}^*$, then $I_\delta(f)$ is absolutely convergent when $0 < \mathrm{Re}(s) < \mathtt{k}$. It vanishes unless $t \in \frac{N}{\mathtt{n}}\mathbf{Z}$. For such t, write $t = \frac{N}{\mathtt{n}}b$. Then*

$$I_\delta(f) = \frac{(4\pi r)^{\mathtt{k}-1}\psi(N)\mathtt{n}^{\mathtt{k}/2}}{(\mathtt{k}-2)!\,e^{i\pi s/2}e^{2\pi r}N^s}b^{s-\mathtt{k}}{}_1f_1(s;\mathtt{k};\tfrac{2\pi i r\mathtt{n}}{Nb})\sum_{\substack{d|b\\ \gcd(b/d,Nd)|\gcd(r,\mathtt{n})}}\frac{\gcd(b/d,Nd)}{d^{2s-\mathtt{k}}\omega'(b/d)}e^{-\frac{2\pi i r\ell_0}{b/d}},$$

where $\ell_0 \in \mathbf{Z}$ is any integer satisfying $\ell_0(Nd) \equiv \mathtt{n} \bmod (b/d)$, and

$${}_1f_1(s;\mathtt{k};w) = \frac{\Gamma(s)\Gamma(\mathtt{k}-s)}{\Gamma(\mathtt{k})}{}_1\mathrm{F}_1(s;\mathtt{k};w).$$

When $b < 0$, we take $b^{s-\mathtt{k}} = |b|^{s-\mathtt{k}}e^{i\pi(s-\mathtt{k})}$.

Proof. The absolute convergence will be proven in Prop. 3.9 below. We can factorize the integral as $I_\delta(f) = I_\delta(f)_\infty I_\delta(f)_{\mathrm{fin}}$. First we compute

$$I_\delta(f)_{\mathrm{fin}} = \int_{\mathbf{A}_{\mathrm{fin}}^*}\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} yx & 1-tx \\ -y & t\end{pmatrix})\theta_{\mathrm{fin}}(rx)dx|y|^{s-\mathtt{k}/2}d^*y.$$

Suppose $f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1-tx \\ -y & t\end{smallmatrix}\right)) \neq 0$. Then there exists $m \in \mathbf{Q}^+$ such that

$$\begin{pmatrix} myx & m-mtx \\ -my & mt\end{pmatrix} \in M(\mathtt{n}, N).$$

This means:

(i) $my \in N\widehat{\mathbf{Z}}$ (iv) $mxy \in \widehat{\mathbf{Z}}$
(ii) $m^2y \in \mathtt{n}\widehat{\mathbf{Z}}^*$ (v) $m - mtx \in \widehat{\mathbf{Z}}$.
(iii) $mt \in \widehat{\mathbf{Z}}$

The first two conditions imply that $m = \frac{\mathtt{n}}{Nd}$ for some integer $d > 0$, and that $y \in \frac{N^2d^2}{\mathtt{n}}\widehat{\mathbf{Z}}^*$. By the third condition, $t \in \frac{Nd}{\mathtt{n}}\widehat{\mathbf{Z}}$, or equivalently, $t \in \frac{N}{\mathtt{n}}\mathbf{Z}$ and $d|\frac{\mathtt{n}}{N}t$. This proves the first assertion. Condition (iv) is now equivalent to $x \in \frac{1}{Nd}\widehat{\mathbf{Z}}$. Conversely, if m, y, t, x are given in this way, they will satisfy (i)-(iv). Thus we have

$$I_\delta(f)_{\text{fin}} = \sum_{d|\frac{\mathtt{n}}{N}t} \frac{\mathtt{n}^{s-\mathtt{k}/2}}{(Nd)^{2s-\mathtt{k}}} \int_{\frac{N^2d^2}{\mathtt{n}}\widehat{\mathbf{Z}}^*} \int_{\frac{1}{Nd}\widehat{\mathbf{Z}}} f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1-tx \\ -y & t \end{smallmatrix}\right))\theta_{\text{fin}}(rx)dx\, d^*y.$$

Write $t = \frac{N}{\mathtt{n}}b$ for nonzero $b \in d\mathbf{Z}$. Then $mt = b/d$, so the fifth condition is equivalent to $x \in \frac{\mathtt{n}}{Nb} + \frac{d}{b}\widehat{\mathbf{Z}}$. Thus the inner integral is taken over

$$x \in \frac{1}{Nd}\widehat{\mathbf{Z}} \cap (\frac{\mathtt{n}}{Nb} + \frac{d}{b}\widehat{\mathbf{Z}}).$$

By Lemma 3.6 (multiply the above through by Nb), this set is nonempty if and only if $\gcd(b/d, Nd)|\mathtt{n}$, in which case it is equal to $\frac{1}{Nd}c_0 + \frac{1}{\gcd(b/d,Nd)}\widehat{\mathbf{Z}}$, where c_0 is any solution to $(b/d)c_0 \equiv \mathtt{n} \bmod Nd$.

Note that $\gcd(b/d, Nd)|\mathtt{n}$ implies that b/d is prime to N. Therefore the value of $f^{\mathtt{n}}$ in the integrand is $\frac{\psi(N)}{\omega'(b/d)}$. Thus

$$I_\delta(f)_{\text{fin}} = \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\mathtt{n}}} \frac{\mathtt{n}^{s-\mathtt{k}/2}}{(Nd)^{2s-\mathtt{k}}} \frac{\psi(N)}{\omega'(b/d)} \int_{\frac{1}{Nd}c_0 + \frac{1}{\gcd(b/d,Nd)}\widehat{\mathbf{Z}}} \theta_{\text{fin}}(rx)dx$$

$$(9) \qquad = \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\mathtt{n}}} \frac{\mathtt{n}^{s-\mathtt{k}/2}}{(Nd)^{2s-\mathtt{k}}} \frac{\psi(N)}{\omega'(b/d)} \theta_{\text{fin}}(\frac{rc_0}{Nd}) \int_{\frac{1}{\gcd(b/d,Nd)}\widehat{\mathbf{Z}}} \theta_{\text{fin}}(rx)dx$$

$$= \frac{\psi(N)\mathtt{n}^{s-\mathtt{k}/2}}{N^{2s-\mathtt{k}}} \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\gcd(r,\mathtt{n})}} \frac{\gcd(b/d,Nd)}{d^{2s-\mathtt{k}}\omega'(b/d)} e^{2\pi i r c_0/Nd}.$$

For the archimedean part, the inner integral is

$$\int_{\mathbf{R}} f_\infty(\left(\begin{smallmatrix} yx & 1-tx \\ -y & t \end{smallmatrix}\right))e^{-2\pi i r x}dx$$
$$= \frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}y^{\mathtt{k}/2} \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(tx - 1 - y + (yx+t)i)^{\mathtt{k}}}dx$$
$$= \frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}y^{\mathtt{k}/2}(t+iy)^{-\mathtt{k}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x - \frac{1+y-it}{i(y-it)})^{\mathtt{k}}}dx.$$

The integrand has a pole at $x = -i(1 + \frac{1}{y-it})$ in the lower half-plane. Using a clockwise lower semicircular contour integral, this is

$$= -\frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}\frac{2\pi i}{(\mathtt{k}-1)!}(-2\pi i r)^{\mathtt{k}-1}y^{\mathtt{k}/2}(i)^{-\mathtt{k}}(y-it)^{-\mathtt{k}}e^{-2\pi r(1+\frac{1}{y-it})}.$$

Thus

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}e^{2\pi r}} \int_0^\infty y^{s-1}(y-it)^{-\mathtt{k}}e^{-2\pi r/(y-it)}dy.$$

This has an essential singularity at $y = it$. We define y^{s-1} as a holomorphic function of y by taking the principal value of $\log y$ on the positive real axis, and making a branch cut along the positive imaginary axis if $t > 0$ or the negative imaginary axis if $t < 0$. Now pulling out t and making a change of variables, we get

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1} t^{s-\mathtt{k}}}{(\mathtt{k}-2)!\, i^{\mathtt{k}} e^{2\pi r}} \int_0^{\pm\infty} y^{s-1}(y-i)^{-\mathtt{k}} e^{-2\pi r/t(y-i)} dy,$$

where the sign in the upper limit is the sign of t, and by our choice of branch, $t^{s-\mathtt{k}} = |t|^{s-\mathtt{k}} e^{i\pi(s-\mathtt{k})}$ if $t < 0$. In the notation of the next lemma below, the integral is $\mathtt{G}(s, \mathtt{k}, r/t)$. By the result of the lemma and setting $t = Nb/\mathtt{n}$, this gives

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1} N^{s-\mathtt{k}}}{(\mathtt{k}-2)!\, e^{i\pi s/2} e^{2\pi r} \mathtt{n}^{s-\mathtt{k}}} \frac{b^{s-\mathtt{k}} {}_1f_1(s; \mathtt{k}; 2\pi i r\mathtt{n}/Nb)}{e^{2\pi i r\mathtt{n}/Nb}}.$$

When we multiply this by $I_\delta(f)_{\mathrm{fin}}$, we can combine the terms

$$e^{-2\pi i r\mathtt{n}/Nb} e^{2\pi i r c_0/Nd} = e^{2\pi i r(c_0(b/d)-\mathtt{n})/Nb}.$$

Writing $c_0(b/d) - \mathtt{n} = -Nd\ell_0$ for some $\ell_0 \in \mathbf{Z}$, we have $Nd\ell_0 \equiv \mathtt{n} \bmod (b/d)$, and the above is equal to $e^{-2\pi i r\ell_0/(b/d)}$. The result now follows. □

Lemma 3.8. *For $s, w \in \mathbf{C}$ and $\mathtt{k} \in \mathbf{Z}^+$, define*

$$\mathtt{G}(s, \mathtt{k}, w) = \int_0^\infty y^{s-1}(y-i)^{-\mathtt{k}} e^{-2\pi w/(y-i)} dy.$$

This function converges absolutely for $0 < \mathrm{Re}(s) < \mathtt{k}$. On this strip we can represent $\mathtt{G}(s, \mathtt{k}, w)$ in terms of the confluent hypergeometric function:

$$\mathtt{G}(s, \mathtt{k}, w) = i^{\mathtt{k}} e^{-i\pi s/2} e^{-2\pi i w} \frac{\Gamma(s)\Gamma(\mathtt{k}-s)}{\Gamma(\mathtt{k})}\, {}_1\mathrm{F}_1(s; \mathtt{k}; 2\pi i w).$$

Furthermore, the integral defining $\mathtt{G}$ is unchanged if we replace ∞ by $-\infty$.

Proof. Let $t = 1 + \frac{i}{y-i}$, so that $y - i = \frac{-i}{1-t}$. This linear fractional transformation takes the positive real axis to the upper semicircle C of radius $1/2$ centered at $z = 1/2$. Then $dy = \frac{-i}{(1-t)^2} dt$ and

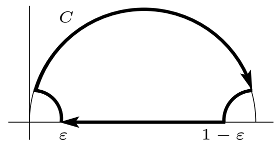
$$\mathtt{G}(s, \mathtt{k}, w) = \int_C \left(\frac{-it}{1-t}\right)^{s-1} \left(\frac{-i}{1-t}\right)^{-\mathtt{k}} e^{2\pi i w(t-1)} \frac{-i}{(1-t)^2} dt.$$

We define $y^{s-1} = e^{(s-1)\log y}$ by taking the principal value of $\log y$ for $y > 0$, and making a cut along the positive imaginary axis in the y-plane. This cut corresponds in the t-plane to cuts on the real axis from 0 to $-\infty$ and from 1 to ∞. We choose $\log(-i) = -i\pi/2$, and choose the principal branches of $\log(t)$ and $\log(1-t)$. Then for $t \in (0, 1)$, $-3\pi/2 < \arg(-it/(1-t)) < \pi/2$ and therefore these choices are compatible with the choice of $\log y$. Now

$$\mathtt{G}(s, \mathtt{k}, w) = e^{-is\pi/2}(-i)^{-\mathtt{k}} e^{-2\pi i w} \int_C e^{2\pi i w t} t^{s-1} (1-t)^{\mathtt{k}-s-1} dt.$$

The integrand is holomorphic in t and single-valued in the cut plane, and by

Cauchy's theorem, its integral around the following contour vanishes:



Using the fact that $0 < \mathrm{Re}(s) < \mathtt{k}$, it is straightforward to show that the contribution along the small arcs goes to 0 as $\varepsilon \to 0$. It follows that the integral along C can instead be taken along the real axis, so

$$\begin{aligned}\mathtt{G}(s,\mathtt{k},w) &= i^{\mathtt{k}} e^{-i\pi s/2} e^{-2\pi i w} \int_0^1 e^{2\pi i w t} t^{s-1} (1-t)^{\mathtt{k}-s-1} dt. \\ &= i^{\mathtt{k}} e^{-i\pi s/2} e^{-2\pi i w} \frac{\Gamma(\mathtt{k}-s)\Gamma(s)}{\Gamma(\mathtt{k})} {}_1\mathrm{F}_1(s;\mathtt{k};2\pi i w)\end{aligned}$$

by (8). If the upper limit of $\mathtt{G}$ is replaced by $-\infty$, then t will instead traverse the lower semicircle $\overline{C}$ from 0 to 1, which can likewise be moved to the real axis. In fact a more general path independence property can be proven in a similar way. □

3.3. **Proof of Proposition 3.3.** For each δ, we set

$$I_\delta^{abs}(f) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} \left| f(\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right) \delta \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)) \overline{\theta_r(x)} |y|^{s-\mathtt{k}/2} \right| dx\, d^*y.$$

Because $f^{\mathtt{n}}$ is compactly supported modulo the center and bounded by $\psi(N)$, the finite part $I_\delta^{abs}(f)_{\mathrm{fin}}$ converges for all s to a value depending on δ. Thus we primarily need to consider the infinite part

$$I_\delta^{abs}(f)_\infty = \int_0^\infty \int_{-\infty}^\infty \left| f_\infty(\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right) \delta \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)) \right| dx\, y^{\mathrm{Re}(s)-\mathtt{k}/2-1} dy.$$

We will repeatedly use the fact that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{R})^+$,

$$|f_\infty(g)| = \frac{\mathtt{k}-1}{4\pi} \frac{\det(g)^{\mathtt{k}/2} 2^{\mathtt{k}}}{(a^2+b^2+c^2+d^2+2\det(g))^{\mathtt{k}/2}}. \tag{10}$$

This follows easily from the explicit formula for f_∞.

Proposition 3.9. *Let* $\delta_t = \left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right)$ *for* $t \in \mathbf{Q}^*$. *Then if* $0 < \mathrm{Re}(s) < \mathtt{k}$,

(a) $I_{\delta_t}^{abs}(f) < \infty$
(b) $I_{\delta_t}^{abs}(f)_\infty \ll |t|^{\mathrm{Re}(s)-\mathtt{k}}$.

Furthermore, if $1 < \mathrm{Re}(s) < \mathtt{k}-1$, *then*

(c) $\sum_{t \in \mathbf{Q}^*} I_{\delta_t}^{abs}(f) < \infty$.

Proof. We need to estimate the expression

$$\int_0^\infty \int_{-\infty}^\infty \left| f_\infty(\begin{pmatrix} yx & 1-tx \\ -y & t \end{pmatrix}) \right| dx\, y^{\mathrm{Re}(s)-\mathtt{k}/2-1} dy.$$

By (10), the inner integral is

$$\begin{aligned}&\ll y^{\mathtt{k}/2} \int_{-\infty}^\infty \frac{dx}{(y^2x^2+y^2+t^2+(1-tx)^2+2y)^{\mathtt{k}/2}} \\ &= y^{\mathtt{k}/2}(t^2+y^2)^{-\mathtt{k}/2} \int_{-\infty}^\infty \frac{dx}{(x^2 - \frac{2t}{t^2+y^2}x + \frac{1+t^2+y^2+2y}{t^2+y^2})^{\mathtt{k}/2}}\end{aligned}$$

We will show that the integral is bounded, independently of y and t. Completing the square, the integral is equal to

$$\int_{-\infty}^{\infty} \frac{dx}{((x-\frac{t}{t^2+y^2})^2+\frac{(1+y)^2+t^2}{t^2+y^2}-\frac{t^2}{(t^2+y^2)^2})^{\mathtt{k}/2}} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+\frac{(1+2y+y^2+t^2)(t^2+y^2)-t^2}{(t^2+y^2)^2})^{\mathtt{k}/2}}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+\frac{(t^2+y^2+y)^2}{(t^2+y^2)^2})^{\mathtt{k}/2}} < \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{\mathtt{k}/2}} < \infty.$$

Therefore writing $s=\sigma+i\tau$,

$$I_{\delta_t}^{abs}(f)_\infty \ll \int_0^\infty y^{\sigma-1}(t^2+y^2)^{-\mathtt{k}/2}dy.$$

For convergence as $y \to 0$, we need $\sigma-1>-1$, i.e. $\sigma>0$. For convergence as $y\to\infty$, we need $\sigma-1-\mathtt{k}<-1$, i.e. $\sigma<\mathtt{k}$. This proves the absolute convergence of $I_{\delta_t}(f)$ on the given strip.

In order to sum over t, we need to bound the above integral in terms of t. We have

$$I_{\delta_t}^{abs}(f)_\infty \ll \int_0^\infty y^{\sigma-1}|t|^{-\mathtt{k}}(1+\frac{y^2}{t^2})^{-\mathtt{k}/2}dy$$
$$= |t|^{-\mathtt{k}}\int_0^\infty \left(\frac{y^2}{t^2}\right)^{\frac{\sigma}{2}} |t|^\sigma (1+\frac{y^2}{t^2})^{-\mathtt{k}/2}d^*y.$$

Letting $u=(y/t)^2$ so $d^*u = 2d^*y$, the above is

$$= \frac{1}{2}|t|^{\sigma-\mathtt{k}}\int_0^\infty \frac{u^{\frac{\sigma}{2}-1}}{(1+u)^{\mathtt{k}/2}}du,$$

which proves the second assertion since $\mathtt{k}>\sigma>0$. (As an aside, this last integral equals $B(\frac{\sigma}{2},\frac{\mathtt{k}-\sigma}{2})$ where $B(n,m)=\Gamma(n)\Gamma(m)/\Gamma(n+m)$ is the Beta function.)

As in the proof of Proposition 3.7, $I_{\delta_t}^{abs}(f)_{\text{fin}}$ vanishes unless $t=\frac{N}{\mathtt{n}}b$ for some $b\in\mathbf{Z}-\{0\}$. By (9), we see that

$$I_{\delta_t}^{abs}(f)_{\text{fin}} \ll \frac{\mathtt{n}^{\sigma-\mathtt{k}/2}\psi(N)}{N^{2\sigma-\mathtt{k}}}\sum_{d|b} d^{-2\sigma+\mathtt{k}}.$$

If $\sigma>\mathtt{k}/2$, then $d^{-2\sigma+\mathtt{k}}\le 1$. If $\sigma\le\mathtt{k}/2$, then $d^{-2\sigma+\mathtt{k}}\le|b|^{-2\sigma+\mathtt{k}}$. The number of divisors of b is $\ll b^\varepsilon$ for any $\varepsilon>0$. Since $I_{\delta_t}^{abs}(f)_\infty$ contributes $|b|^{\sigma-\mathtt{k}}$, we have

$$\sum_{t\in\mathbf{Q}^*} I_{\delta_t}^{abs}(f) \ll \begin{cases} \sum_{b\in\mathbf{Z}-\{0\}}|b|^{-\sigma+\varepsilon} & \text{if } \sigma\le\mathtt{k}/2 \\ \sum_{b\in\mathbf{Z}-\{0\}}|b|^{\sigma-\mathtt{k}+\varepsilon} & \text{if } \sigma>\mathtt{k}/2. \end{cases} \tag{11}$$

Hence $\sum_t I_{\delta_t}^{abs}(f)<\infty$ as long as $1<\sigma<\mathtt{k}-1$. □

The following will complete the proof of Proposition 3.3.

Proposition 3.10. *For $\delta=1$, $I_1^{abs}(f)<\infty$ provided*

$$0<\mathrm{Re}(s)<\mathtt{k}-1.$$

For $\delta=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, $I_\delta^{abs}(f)<\infty$ provided

$$1<\mathrm{Re}(s)<\mathtt{k}.$$

Proof. For any $a > 0$, a change of variables gives

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^{\mathtt{k}/2}} = a^{-\mathtt{k}+1} \int_{-\infty}^{\infty} \frac{du}{(u^2+1)^{\mathtt{k}/2}}. \tag{12}$$

We again write $s = \sigma + i\tau$. When $\delta = 1$, using (10) we have

$$I_1^{abs}(f) \ll \int_0^{\infty} \int_{-\infty}^{\infty} \frac{y^{\sigma-1}}{(x^2+y^2+2y+1)^{\mathtt{k}/2}} dx\, dy.$$

By (12), this is

$$\ll \int_0^{\infty} y^{\sigma-1}(y+1)^{-\mathtt{k}+1} dy.$$

This converges precisely when $0 < \sigma < \mathtt{k} - 1$.

Similarly, for $\delta = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix} \right)$,

$$\begin{aligned} I_\delta^{abs}(f) &\ll \int_0^{\infty} \int_{-\infty}^{\infty} \frac{y^{\sigma-1}}{(x^2y^2+y^2+2y+1)^{\mathtt{k}/2}} dx\, dy \\ &= \int_0^{\infty} y^{\sigma-1-\mathtt{k}} \int_{-\infty}^{\infty} \frac{dx}{(x^2+(1+\frac{1}{y})^2)^{\mathtt{k}/2}}\, dy \\ &\ll \int_0^{\infty} y^{\sigma-\mathtt{k}-1}(1+y^{-1})^{-\mathtt{k}+1} dy. \end{aligned}$$

As $y \to 0$, we need $\sigma - \mathtt{k} - 1 + \mathtt{k} - 1 > -1$, i.e. $\sigma > 1$. As $y \to \infty$, we need $\sigma - \mathtt{k} - 1 < -1$, i.e. $\sigma < \mathtt{k}$. This proves the proposition. □

3.4. **Proof of Theorem 1.1.** We have now proven that the geometric side converges absolutely when $1 < \mathrm{Re}(s) < \mathtt{k} - 1$, and therefore it is equal to the spectral side on this strip. When we sum the contribution of Prop. 3.7 over all $b \neq 0$, we set $a = b/d$ so that $b = ad$. Then

$$\begin{aligned} &\sum_{b\neq 0} b^{s-\mathtt{k}} {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r \mathtt{n}}{Nb}) \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\gcd(r,\mathtt{n})}} \frac{\gcd(b/d,Nd)}{d^{2s-\mathtt{k}}\omega'(b/d)} e^{-\frac{2\pi i r \ell_0}{b/d}} \\ &\qquad = \sum_{\substack{a\neq 0, d>0 \\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}} \frac{a^{s-\mathtt{k}} d^{-s} \gcd(a,Nd)}{\omega'(a) e^{2\pi i r \ell_0/a}}\, {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r \mathtt{n}}{Nad}). \end{aligned}$$

The theorem now follows immediately upon equating the two sides of the trace formula and dividing through by $e^{-2\pi r}\mathtt{n}^{1-\mathtt{k}/2}$.

4. Estimates and Examples

4.1. **Asymptotic behavior.** For two functions A, B, we write $A \sim B$ to mean that $A/B \to 1$ in a limiting sense which will be clear from the context. For example, by Stirling's approximation we have the following:

$$\Gamma(z+b) \sim \sqrt{2\pi} e^{-z} z^{z+b-1/2} \qquad (z \to \infty, |\arg z| < \pi) \tag{13}$$

([AS], 6.1.39). The $\sim$ notation here depends on b; i.e. given $\varepsilon > 0$ there is a constant $N(b) > 0$ such that the quotient is within ε of 1 whenever $|z| > N(b)$.

We now estimate each term of Theorem 1.1 as $\mathtt{k} \to \infty$. It will turn out that the first two terms are dominant, provided their sum does not vanish. In order to ensure nonvanishing of $\sum_{m|\gcd(\mathtt{n},r)} m^{2s-\mathtt{k}+1}/\omega'(m)$, we will assume for simplicity that $\gcd(\mathtt{n}, r) = 1$. However in general one can prove that this sum can only vanish on the left edge of the critical strip, i.e. on the line $\mathrm{Re}(s) = \frac{\mathtt{k}-1}{2}$.

Proposition 4.1. *Let $s = \mathtt{k}/2 + \alpha + i\tau$, with $1 < \mathtt{k}/2 + \alpha < \mathtt{k} - 1$. Assume $\gcd(\mathtt{n}, r) = 1$. Then as $\mathtt{k} \to \infty$ the identity term in Theorem 1.1 satisfies*

$$\left|\frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!}\right| \sim \frac{2\sqrt{\pi}\,\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}/2-\alpha-1}\mathtt{k}^{\mathtt{k}/2+\alpha-1/2}}{(\mathtt{k}-2)!\,e^{\mathtt{k}/2}} \sim \sqrt{2}\psi(N)e^{\alpha+1}\left(\frac{4\pi r\mathtt{n}e}{\mathtt{k}}\right)^{\mathtt{k}/2-\alpha-1}. \tag{14}$$

If $N = 1$, then as $\mathtt{k} \to \infty$ the second term in Theorem 1.1 satisfies

$$\left|\frac{2^{\mathtt{k}-1}\Gamma(\mathtt{k}-s)(2\pi r\mathtt{n})^{s-1}}{(\mathtt{k}-2)!\,i^{\mathtt{k}}}\right| \sim \frac{2\sqrt{\pi}(4\pi r\mathtt{n})^{\mathtt{k}/2+\alpha-1}\mathtt{k}^{\mathtt{k}/2-\alpha-1/2}}{(\mathtt{k}-2)!\,e^{\mathtt{k}/2}} \sim \sqrt{2}e^{-\alpha+1}\left(\frac{4\pi r\mathtt{n}e}{\mathtt{k}}\right)^{\mathtt{k}/2+\alpha-1}. \tag{15}$$

Remark: The $\sim$ notation here depends on $\alpha + i\tau$ as discussed after (13).

Proof. Using (13), the lefthand side of (14) is

$$= \frac{\psi(N)2^{\mathtt{k}-1}|\Gamma(\mathtt{k}/2+\alpha+i\tau)|(2\pi r\mathtt{n})^{\mathtt{k}/2-\alpha-1}}{(\mathtt{k}-2)!} \sim \frac{\psi(N)2^{-1}2^{\mathtt{k}}(2\pi r\mathtt{n})^{\mathtt{k}/2-\alpha-1}\sqrt{2\pi}e^{-\mathtt{k}/2}(\mathtt{k}/2)^{\mathtt{k}/2+\alpha-1/2}}{(\mathtt{k}-2)!}.$$

For the second line of (14) we substitute $(\mathtt{k}-2)! = \Gamma(\mathtt{k}-1) \sim \sqrt{2\pi}e^{-\mathtt{k}}\mathtt{k}^{\mathtt{k}-1-1/2}$. The second estimate is similar, as the lefthand side of (15) is

$$\sim \frac{2^{-1}2^{\mathtt{k}}(2\pi r\mathtt{n})^{\mathtt{k}/2+\alpha-1}e^{-\mathtt{k}/2}(\mathtt{k}/2)^{\mathtt{k}/2-\alpha-1/2}}{(\mathtt{k}-2)!}.$$

□

We now show that the third term in Theorem 1.1 decays much more rapidly in comparison with the first terms as $\mathtt{k} \to \infty$. We can rewrite it as a sum over $a, d > 0$. Note that $\omega'(-a) = (-1)^{\mathtt{k}}\omega'(a)$. Thus the third term is equal to

$$\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\,e^{i\pi s/2}} \sum_{\substack{a,d>0 \\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}} \Big[a^{s-\mathtt{k}}d^{-s}{}_1f_1(s;\mathtt{k};\tfrac{2\pi ir\mathtt{n}}{Nad})e^{-2\pi ir\ell_0/a} + e^{i\pi s}a^{s-\mathtt{k}}d^{-s}{}_1f_1(s;\mathtt{k};-\tfrac{2\pi ir\mathtt{n}}{Nad})e^{2\pi ir\ell_0/a}\Big]\frac{\gcd(a,Nd)}{\omega'(a)}, \tag{16}$$

where ℓ_0 is any integer satisfying $\ell_0 Nd \equiv \mathtt{n} \bmod a$. Write $s = \sigma + i\tau$. If w is real,

$$|{}_1f_1(s;\mathtt{k};2\pi iw)| = \left|\int_0^1 e^{2\pi iwt}t^{s-1}(1-t)^{\mathtt{k}-s-1}dt\right| \le \int_0^1 t^{\sigma-1}(1-t)^{\mathtt{k}-\sigma-1}dt = B(\sigma,\mathtt{k}-\sigma) \tag{17}$$

for the Beta function B. Furthermore, $|e^{i\pi s/2}| = e^{-\pi\tau/2}$. Thus the absolute value of (16) is

$$\le \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}\gcd(r,\mathtt{n})B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!}e^{\pi\tau/2}\sum_{a,d>0} a^{-(\mathtt{k}-\sigma)}d^{-\sigma}\left|1+e^{i\pi s}\right|.$$

Note that $|1+e^{i\pi s}| \le (1+e^{-\pi\tau})$. Pulling this out of the sum, we obtain $(e^{\pi\tau/2} + e^{-\pi\tau/2}) = 2\cosh(\tau\pi/2)$, and we immediately arrive at the following.

Proposition 4.2. *Write $s = \sigma + i\tau$ for $1 < \sigma < \mathtt{k}-1$. Then the absolute value of the last term* (16) *of Theorem 1.1 is*

$$\le \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}\gcd(r,\mathtt{n})B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!} 2\cosh(\tau\pi/2)\zeta(\mathtt{k}-\sigma)\zeta(\sigma)$$

for the Beta function B and the Riemann zeta function ζ.

We remark that when $1 < \mathrm{Re}(s) < \mathtt{k}-1$ as is the case here, the integrand in (17) is smaller than 1 so $0 < B(\sigma,\mathtt{k}-\sigma) < 1$.

If we restrict s to the critical strip $\frac{\mathtt{k}-1}{2} < \mathrm{Re}(s) < \frac{\mathtt{k}+1}{2}$, then both zeta values approach 1 as $\mathtt{k}\to\infty$. Therefore we see that if $N > 1$, the identity term is dominant as $\mathtt{k}\to\infty$. If $N = 1$, then $I_1(f)$ is the main term when $\sigma > \mathtt{k}/2$, while $I_{\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)}(f)$ is the main term when $\sigma < \mathtt{k}/2$.

Corollary 1.3 now follows easily. In fact we can make it effective. Assume $N > 1$, $\mathtt{k} > 3$ and $\gcd(\mathtt{n}, r) = 1$. Let

$$F(s) = \frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!}$$

denote the first term of the geometric side of Theorem 1.1, and let $T(s)$ denote the other term, given in (16). Clearly the average of L-values is nonzero whenever $|T(s)| < |F(s)|$. By Prop. 4.2, this holds whenever

$$\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!} 2\cosh(\pi\tau/2)\zeta(\mathtt{k}-\sigma)\zeta(\sigma) < \frac{\psi(N)2^{\mathtt{k}-1}|\Gamma(s)|(2\pi r\mathtt{n})^{\mathtt{k}-\sigma-1}}{(\mathtt{k}-2)!}.$$

Using $B(\sigma,\mathtt{k}-\sigma) = \Gamma(\sigma)\Gamma(\mathtt{k}-\sigma)/(\mathtt{k}-1)!$, the above is equivalent to

$$2\cosh(\tau\pi/2) < \left(\frac{N}{2\pi r\mathtt{n}}\right)^\sigma \frac{(\mathtt{k}-1)!\,|\Gamma(s)|}{\zeta(\mathtt{k}-\sigma)\zeta(\sigma)\Gamma(\mathtt{k}-\sigma)\Gamma(\sigma)}. \tag{18}$$

Lemma 4.3. *For any $s = \sigma + i\tau$ with $\sigma > 1$,*

$$\left|\frac{\Gamma(s)}{\Gamma(\sigma)}\right| \ge e^{-\tau\arg(s-1/2)} \left(\frac{\sigma - 1/2 - \frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\sigma - 1/2 + \frac{\ln(2)}{\pi\sqrt{2\pi e}}}\right).$$

Proof. This follows immediately from the following approximation due to Spouge:

$$\Gamma(s) = \sqrt{2\pi}(s-1/2)^{s-1/2}e^{-s+1/2}[1+\varepsilon(s)] \qquad (\sigma > 1), \tag{19}$$

where

$$|\varepsilon(s)| < \frac{\frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\sigma - 1/2}$$

([Sp], Theorem 1.3.2). We apply this to $\Gamma(s)$ and $\Gamma(\sigma)$, and use

$$\left|(s-1/2)^{s-1/2}\right| = |s-1/2|^{\sigma-1/2}e^{-\tau\arg(s-1/2)} \ge (\sigma-1/2)^{\sigma-1/2}e^{-\tau\arg(s-1/2)}.$$

□

By the lemma and (18), we see that the average of Theorem 1.1 is nonzero whenever

$$2\cosh(\tau\pi/2)e^{\tau\arg(s-1/2)} < \frac{\left(\frac{N}{2\pi r\mathtt{n}}\right)^{\sigma}(\mathtt{k}-1)!}{\zeta(\mathtt{k}-\sigma)\zeta(\sigma)\Gamma(\mathtt{k}-\sigma)}\left(\frac{\sigma-1/2-\frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\sigma-1/2+\frac{\ln(2)}{\pi\sqrt{2\pi e}}}\right). \tag{20}$$

We remark that since $|\arg(s-1/2)|<\pi/2$, the lefthand side is bounded above by $2\cosh(\tau\pi/2)e^{|\tau|\pi/2}=e^{\pi|\tau|}+1$, which would simplify but weaken the inequality.

Since the lefthand side of (20) increases with $|\tau|$, we obtain the following.

Proposition 4.4. *Suppose $N>1$, $\mathtt{k}>3$, and $\gcd(\mathtt{n},r)=1$. Fix $\tau_0>0$, and let R denote the set of $s=\sigma+i\tau$ with $|\tau|\le\tau_0$ and $\frac{\mathtt{k}-1}{2}\le\sigma\le\frac{\mathtt{k}+1}{2}$. Then the average in Theorem 1.1 is nonzero at every point of R if*

$$2\cosh(\tau_0\pi/2)e^{\tau_0\tan^{-1}(\frac{\tau_0}{\mathtt{k}/2-1})} < \frac{\left(\frac{N}{2\pi r\mathtt{n}}\right)^{\frac{\mathtt{k}\pm1}{2}}(\mathtt{k}-1)!}{\zeta(\frac{\mathtt{k}-1}{2})^2\,\Gamma(\frac{\mathtt{k}+1}{2})}\left(\frac{\frac{\mathtt{k}}{2}-1-\frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\frac{\mathtt{k}}{2}+\frac{\ln(2)}{\pi\sqrt{2\pi e}}}\right). \tag{21}$$

Here we choose $\frac{\mathtt{k}-1}{2}$ if $N>2\pi r\mathtt{n}$, and $\frac{\mathtt{k}+1}{2}$ otherwise.

Because the righthand side of (21) tends to ∞ as $N+\mathtt{k}\to\infty$, Corollary 1.3 follows immediately.

4.2. **Zero-free regions.** We can use Prop. 4.4 to find zero-free regions of certain modular L-functions. The idea is to apply the proposition with $\mathtt{n}=r=1$ when $\dim S_{\mathtt{k}}(N,\omega')=1$, since the average then gives an actual L-value. The exponent of $\frac{N}{2\pi}$ in (21) is $\frac{\mathtt{k}+1}{2}$ unless $N\ge 7$.

Example 4.5. *Let h denote the unique normalized cusp form in $S_{10}(2)$. When $\mathtt{n}=r=1$, $N=2$ and $\mathtt{k}=10$, the righthand side of (21) is 8.97346, and the inequality holds for $\tau_0=1.169259$. Hence the value of $\Lambda(s,h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)|\le 1.169259$.*

Example 4.6. *Let h denote the unique normalized cusp form in $S_8(3)$. Then $\Lambda(s,h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)|\le 1.119308$.*

Example 4.7. *Let h denote the unique normalized cusp form in $S_6(5)$. Then the value of $\Lambda(s,h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)|\le 0.852608$.*

Example 4.8. *According to Stein's Modular Forms Database, there exists a Dirichlet character χ mod 7 (unique up to Galois conjugacy) for which $\dim S_5(7,\chi)=1$. If h is the normalized cusp form, then $\Lambda(s,h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)|\le 0.501352$.*

4.3. **Approximation by partial sums.** In order to estimate the geometric side, we can truncate the last term (16). Let A,D be positive integers. Define the partial sum

$$S_{A,D}=\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\,e^{i\pi s/2}}\sum_{\substack{1\le a\le A,1\le d\le D\\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}}\Big[a^{s-\mathtt{k}}d^{-s}{}_1f_1(s;\mathtt{k};\tfrac{2\pi ir\mathtt{n}}{Nad})e^{-2\pi ir\ell_0/a}$$
$$+e^{i\pi s}a^{s-\mathtt{k}}d^{-s}{}_1f_1(s;\mathtt{k};-\tfrac{2\pi ir\mathtt{n}}{Nad})e^{2\pi ir\ell_0/a}\Big]\frac{\gcd(a,Nd)}{\omega'(a)}$$

where as usual $\ell_0 Nd \equiv \mathtt{n} \bmod a$. The error is given by the tail of the series

$$\Delta_{A,D} = \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\, e^{i\pi s/2}} \sum_{\substack{a,d>0 \\ a>A \text{ or } d>D \\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}} \Big[a^{s-\mathtt{k}} d^{-s} {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r\mathtt{n}}{Nad})e^{-2\pi i r\ell_0/a}$$
$$+ e^{i\pi s} a^{s-\mathtt{k}} d^{-s} {}_1f_1(s;\mathtt{k};-\tfrac{2\pi i r\mathtt{n}}{Nad})e^{2\pi i r\ell_0/a}\Big] \frac{\gcd(a,Nd)}{\omega'(a)}.$$

As in the proof of Prop. 4.2, we have the following bound for the error:

$$|\Delta_{A,D}| \le \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}\gcd(r,\mathtt{n})B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!} 2\cosh(\pi\tau/2) \sum_{\substack{a>A \text{ or } d>D \\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}} a^{-(\mathtt{k}-\sigma)} d^{-\sigma}.$$

We can estimate the error using the following easy lemma.

Lemma 4.9. *For $s = \sigma + i\tau$,*

$$\sum_{\substack{a>A \text{ or } d>D \\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}} a^{-(\mathtt{k}-\sigma)} d^{-\sigma} \le \zeta(\mathtt{k}-\sigma)\zeta(\sigma) - \sum_{a=1}^{A} a^{-(\mathtt{k}-\sigma)} \sum_{d=1}^{D} d^{-\sigma}.$$

4.4. **Computing the τ-function.** As a simple example, consider Ramanujan's $\Delta(z) = \sum_{n=1}^\infty \tau(n)e^{2\pi i n z} \in S_{12}(1)$. Writing

$$\tau(r) = \frac{\tau(r)\Lambda(6,\Delta)/\|\Delta\|^2}{\tau(1)\Lambda(6,\Delta)/\|\Delta\|^2}, \tag{22}$$

we can use the geometric side of Theorem 1.1 to compute the top and bottom. Taking $\mathtt{n} = 1$, let $F(r)$ denote the sum of the first two terms of the formula for $\frac{\tau(r)\Lambda(6,\Delta)}{\|\Delta\|^2}$. We find that

$$F(r) = \frac{2^{12}(2\pi r)^5 5!}{10!}.$$

Let $S_A(r)$ denote the A^{th} partial sum (taking $A = D$ above) of the last term of the formula. Then $\frac{\tau(r)\Lambda(6,\Delta)}{\|\Delta\|^2} \approx F(r) + S_A(r)$ with an error of

$$\le \frac{2(4\pi r)^{11}B(6,6)}{10!}\left[\zeta(6)^2 - \left(\sum_{a=1}^{A}\frac{1}{a^6}\right)^2\right] \tag{23}$$

by Lemma 4.9.

As an illustration, we will compute $\tau(2)$. To estimate the denominator of (22), take $r = 1$ and $A = 1$. This gives

$$\frac{\Lambda(6,\Delta)}{\|\Delta\|^2} \approx \frac{2^{12}(2\pi)^5 5!}{10!} - \frac{(4\pi)^{11}}{10!}\Big[{}_1f_1(6;12;2\pi i) + {}_1f_1(6;12;-2\pi i)\Big] = 1492.55$$

with an error of ≤ 8.584. So the exact value is in the interval $[1483, 1502]$.

For $r = 2$ we need to use $A = 3$ to get a reasonable approximation. We get

$$\frac{\tau(2)\Lambda(6,\Delta)}{\|\Delta\|^2} \approx \frac{2^{12}(4\pi)^5 5!}{10!} - \frac{(8\pi)^{11}}{10!} \sum_{\substack{a,d\in\{1,2,3\} \\ \gcd(a,d)=1}} (ad)^{-6} \Big[{}_1f_1(6;12;\tfrac{4\pi i}{ad})e^{-2\pi i r\ell_0/a}$$
$$+ {}_1f_1(6;12;-\tfrac{4\pi i}{ad})e^{2\pi i r\ell_0/a}\Big]$$
$$= -35769.72.$$

By (23) the error here is

$$\leq \frac{2(8\pi)^{11}B(6,6)}{(10)!}\left(\zeta(6)^2-(1+\frac{1}{2^6}+\frac{1}{3^6})^2\right)=354.008.$$

Thus the exact value is in the interval $[-36124, -35415]$.

Taking the quotient of the estimates, we find that

$$\frac{-36124}{1483}\leq\tau(2)\leq\frac{-35415}{1502},$$

i.e.

$$-24.359\leq\tau(2)\leq-23.578.$$

Because $\tau(2)$ is an integer, it must equal -24.

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